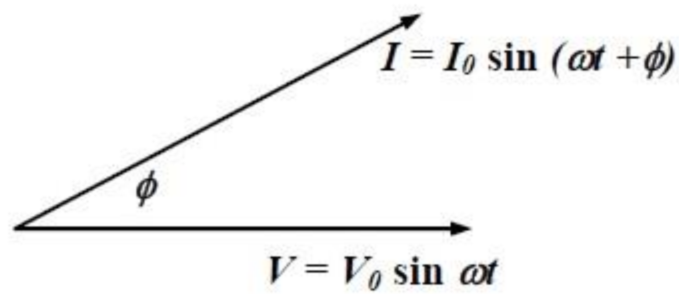
Power in AC Circuits

Let the voltage of an AC circuit be $V = V_0 \sin \omega t$ and current of the form $I = I_0 \sin (\omega t + \phi)$.

The phase difference of the current and the voltage of this circuit is ϕ.



The **work done** at any instant is given by Joule's Law, (P = VI)

$$\mathbf{P = VI = V_0 I_0 \sin \omega t \sin (\omega t + \phi)}$$

The **average value** for a complete cycle is

$$P_{av} = \frac{1}{T}\int_0^T VI \, dt$$

$$= \frac{V_0 I_0}{T}\int_0^T \sin\omega t \, \sin(\omega t + \phi) dt$$

$$= \frac{V_0 I_0}{T}\int_0^T (\sin^2 \omega t \, \cos\phi + \sin\omega t \cos\omega t \, \sin\phi) \, dt$$

$$= \frac{V_0 I_0}{T}\left[\int_0^T \sin^2 \omega t \, \cos\phi \, dt + \int_0^T \sin\omega t \cos\omega t \, \sin\phi \, dt\right]$$

$$= \frac{V_0 I_0}{T}\left[\frac{\cos\phi}{2}\int_0^T (1-\cos 2\omega t)\ \ \mathrm{dt} + \frac{\sin\phi}{2}\int_0^T \sin 2\omega t\ \ \mathrm{dt}\right]$$

$$= \frac{V_0 I_0}{T}\left[\frac{T}{2}\right]\cos\phi$$

$$= \frac{V_0}{\sqrt{2}}\frac{I_0}{\sqrt{2}}\cos\phi$$

$$= V_{rms} I_{rms}\ \cos\phi$$

In DC circuits we can directly multiply voltage by current.

But in AC circuits since the voltage and current are not in phase.

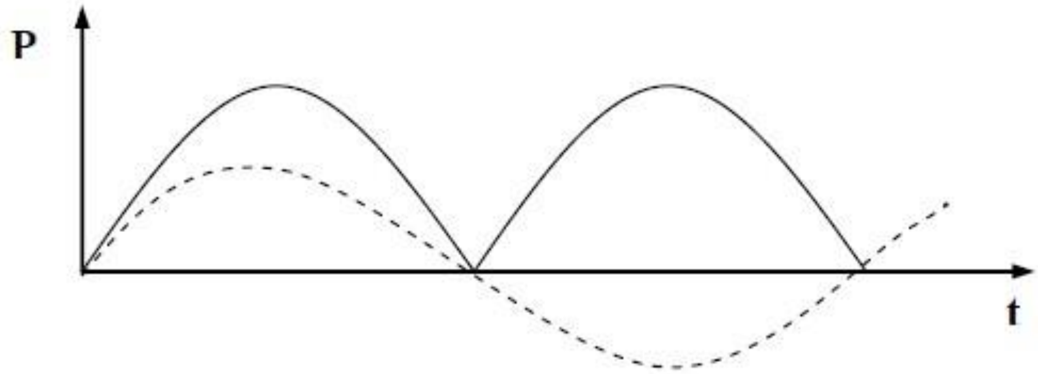
We have to multiply the V_{rms} by the projection of I_{rms} in the direction of voltage.

Cos ϕ is known as the power factor of the circuit.

In AC circuits only with resistance, the current and voltage are in phase.

Therefore $\phi = 0$.

Thus power of such a circuit is V_{rms} x I_{rms}



The variation of power with time over a cycle is shown in figure.

Since $V_\circ = I_\circ R$ the above relation for power can also be written as

$$P_{av} = \left(\frac{I_o}{\sqrt{2}}\right)^2 R = I_{eff}{}^2 R$$

$$= \frac{1}{R}\left(\frac{V_o}{\sqrt{2}}\right)^2 = \frac{(V_{eff})^2}{R}$$

The average power dissipated in the resistance R thus is the same as if a direct current of magnitude equal to the effective value of A.C. were flowing through the resistor.

In AC circuits only with capacitance or inductance $\phi = \pi/2$. Therefore average power consumed by pure inductor or capacitor is zero.

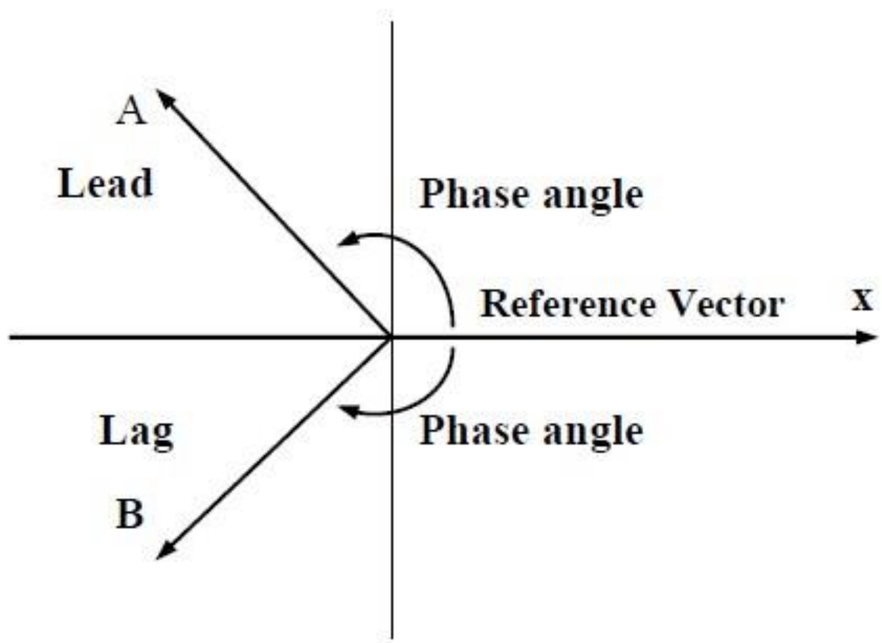
Graphical Representation of Rotating Vectors

Vectors are very useful for expressing the phase difference or phase angle between **AC voltages and currents**.

A.C. vectors are called rotating vectors since they represent sine or cosine quantities and arise from the rotation of the armature of an A.C. generator.

A.C. or rotating vectors show both the amplitude and the phase relationship between sinusoidal voltages and currents.

The length of the vector shows amplitude while the angle between the vectors shows the phase. Also the position of the vectors shows which is leading and which is lagging.



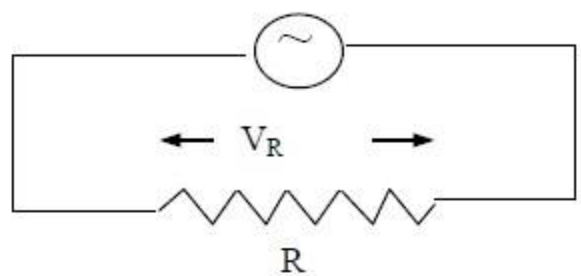
Vector Diagrams of Circuit Elements

Resistive Circuit

Take a resistance R connected to an A.C. supply as shown in figure.

Let *i* be the current through the circuit at any instant.

Then i can be written as $i = i_o \sin \omega t$



The potential difference across R is V_R where $V_R = i_0 R \sin \omega t$

Therefore there is no angular displacement between *i* and V_R so that their phase displacement is zero. **(in phase).**

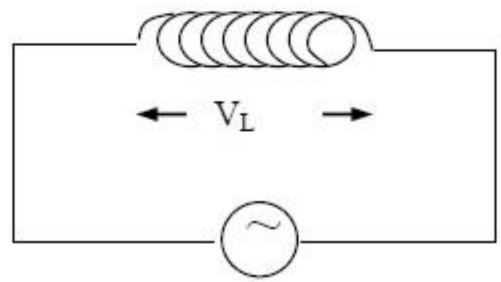
Then the ***vector or phasor diagram*** for V_R can be drawn as shown in the figure

Here current and voltage are in the same direction and the direction of the current is taken as the reference axis.

Take $i_o R = V_o$ Then, $V_R = V_o \sin \omega t$

Inductive Circuit

Consider an inductor of inductance L connected to an A.C supply as shown in figure. We assume that there is no resistance in the circuit.



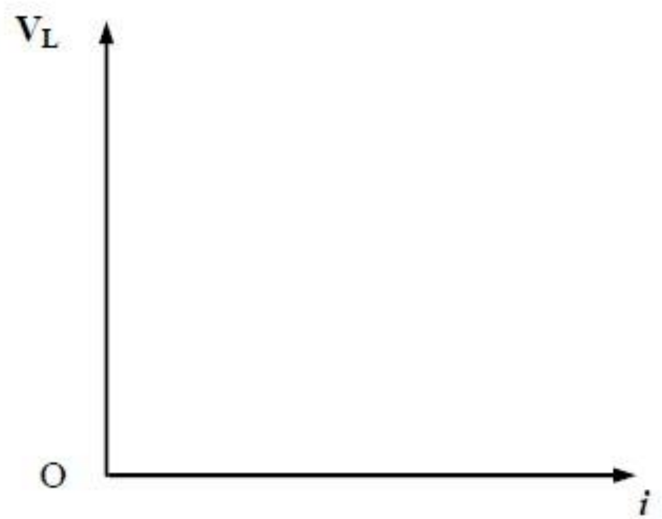
Take the current through the inductor at any instant as $i = i_o \sin \omega t$

The potential difference across L is $V_L = L\frac{di}{dt} = Li_o\omega\cos\omega t$

$$V_L = Li_o\omega\sin\left(\omega t + \frac{\pi}{2}\right)$$

Amplitude of this voltage is $Li_o\omega$ and the voltage is ahead of the current by $\pi/2$ (90°)

Taking the direction of i as the reference axis the *vector or* ***phasor diagram*** may be drawn as in figure.



The length of the vector V_L *is* proportional to the amplitude $i_o L\omega$.

Capacitive Circuit

Let C be the capacitance connected to the A.C. supply. Take the current at a given moment to be *i* which can be written as $i = i_o \sin \omega t$

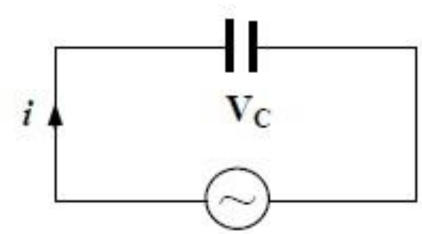
Then, the potential difference across the capacitance is given by V_C,

$$V_C = \frac{q}{C}$$

But, *q* can be written as $q = \int i dt$

Therefore, $V_C = \frac{1}{C}\int i dt$

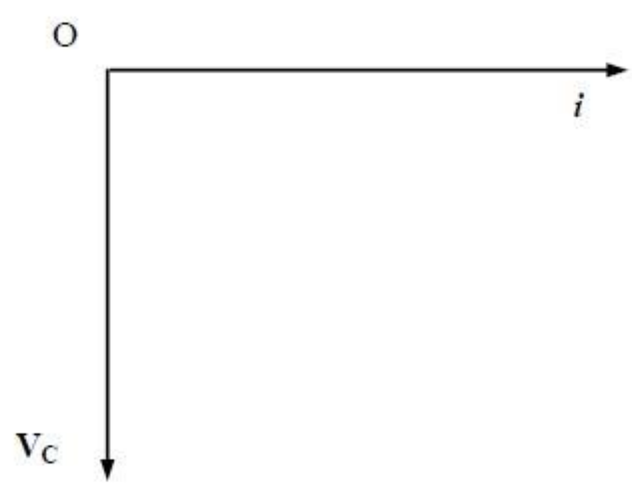
$$V_C = \frac{1}{C}\int i_o \sin \omega t \, dt = -\frac{i_o}{\omega C}\cos \omega t$$



This may be written as

$$V_C = \frac{i_o}{C\omega}\sin\left(\omega t - \frac{\pi}{2}\right)$$

This shows that the potential difference across the capacitor lags behind the current by $\pi/2$ (90°).



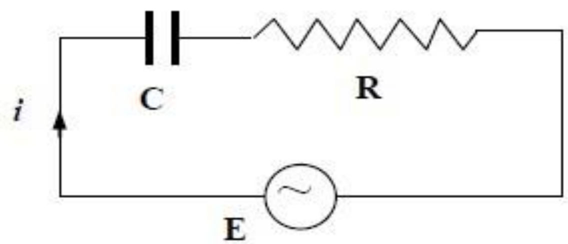
The amplitude or the peak value of V_C is $i_o/C\omega$.

Taking the direction of ***i*** as the reference axis, ***vector or phasor diagram*** can be drawn as follows.

Then the length of vector is proportional to the value of amplitude, $i_o/\omega C$.

C-R Circuit

Take a circuit which contains resistance and capacitance only. They are in series. Let the current through the circuits is given by $\mathrm{i} = \mathrm{i}_{\mathrm{o}} \sin \omega \mathrm{t}$



The voltage difference across the combination is E given by

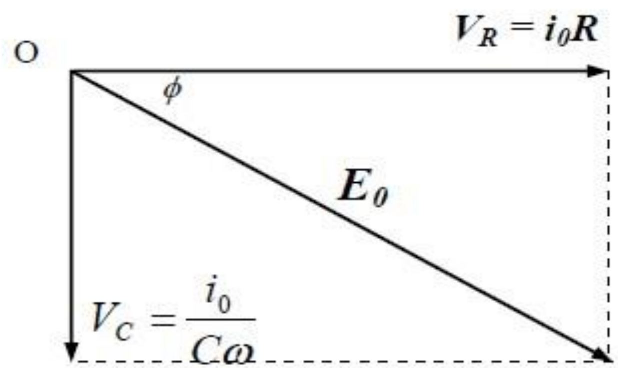
$$E = V_C + V_R$$

Where, V_C and V_R are the potential differences across the capacitance and the resistance.

Then, $E = \dfrac{i_o}{C\omega} \sin\left(\omega t - \pi/2\right) + i_o R \sin \omega t$

The value of the resultant E can be taken by drawing a ***vector or phsor diagram*** as follows.

Take the reference direction as the direction of the current, and draw the diagram as shown taking the phases into account.



This shows that the resultant voltage lags behind the current by an angle ϕ

Where $\tan\phi = \dfrac{i_C / C\omega}{i_o R}$

$$\tan\phi = \frac{1}{RC\omega}$$

The peak value of E is E_0. It is given by,

$$E_o = \sqrt{(i_o R)^2 + \left(\frac{i_o}{C\omega}\right)^2} = i_o \sqrt{R^2 + \frac{1}{C^2\omega^2}}$$

This indicates that the potential difference, E at any instant, is of the form,

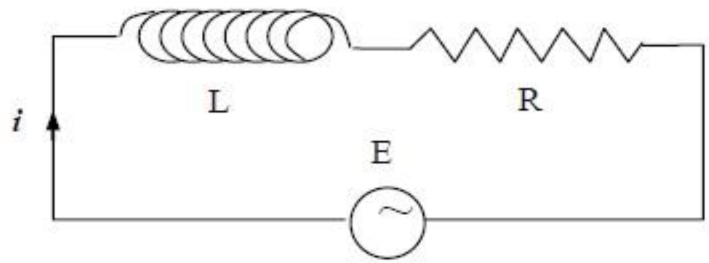
$$E = E_o \sin(\omega t - \phi)$$

Where $$E_o = i_o \sqrt{R^2 + \frac{1}{C^2\omega^2}}$$

and $$\tan\phi = \frac{1}{RC\omega}$$

L-R Circuit

Let the alternating current i be applied to a circuit having resistance and inductance only.



The potential difference E for this circuit is given by,

$E = V_L + V_R$ where V_L and V_R are the potential difference across L and R.

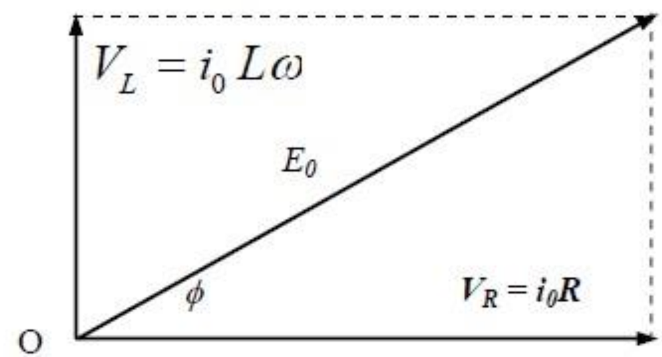
The current i can be written as $i = i_o \sin \omega t$

Then, $E = L\dfrac{di}{dt} + iR$

$$E = Li_o \omega \sin\left(\omega t + \frac{\pi}{2}\right) + i_o R \sin \omega t$$

To get the resultant E the *vector or phasor diagram* is drawn. Then, the maximum value of E is equal to E_0 which is given by

$$E_O = i_o \sqrt{R^2 + L^2\omega^2}$$



and the phase difference of E is ϕ

Where, $\tan \phi = \frac{L\omega}{R}$

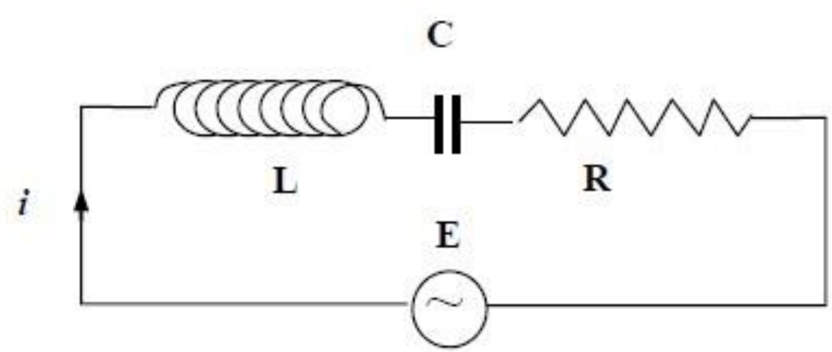
Therefore, at any instant E can be given by the equation.

$$E = E_o \sin(\omega t + \phi)$$

L-C-R Series Circuits

In a series circuit the same current passes through all parts of the circuit and the vector sum of the voltage drops across the components is equal to the impressed voltage E.

The algebraic sum of the voltage drops in an A.C. circuit may exceed the impressed voltage.



Let $i = i_o \sin \omega t$ be the current passing through the circuit.

Then, $$E = V_L + V_C + V_R$$

Since the same current $i = i_o \sin \omega t$ passes through each of the components L, C and R.

$$V_L = L\frac{di}{dt} = i_o L\omega \sin\left(\omega t + \frac{\pi}{2}\right) = i_o X_L \sin\left(\omega t + \pi/2\right)$$

$$V_R = i_o R = i_o R \sin \omega t$$

$$V_C = \frac{\int i dt}{C} = \frac{i_o}{C\omega}\sin\left(\omega t - \pi/2\right) = i_o X_L \sin\left(\omega t - \pi/2\right)$$

***The vector representing inductive reactance (X_L) is plotted upwards above the horizontal axis while the vector representing capacitance reactance (X_C) is plotted downwards from the horizontal axis.**

***The inductive reactance vector leads the resistive of reference vector (on the X-axis) by 90^o while the capacitive reactance vector lags by 90^o.**

$$E = i_o R \sin \omega t + i_o L\omega \sin\left(\omega t + \pi/2\right) + \frac{i_o}{C\omega} \sin\left(\omega t - \pi/2\right)$$

Then, $E_0{}^2 = i^2\left(R^2\right) + i_o{}^2\left(L\omega - \frac{1}{C\omega}\right)^2$

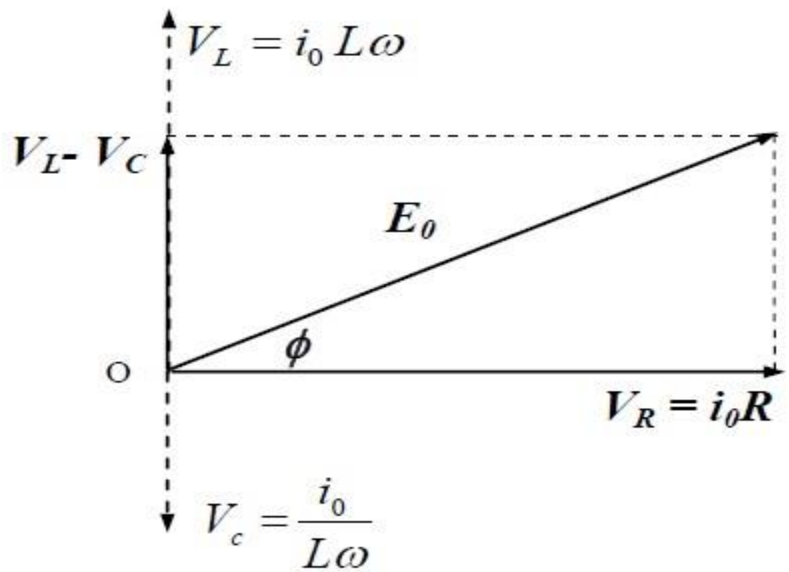
$$E_0 = i_o \sqrt{R^2 + \left(L\omega - \frac{1}{C\omega}\right)^2}$$ and

$$Z = \sqrt{R^2 + \left(L\omega - \frac{1}{C\omega}\right)^2}$$

But $L\omega = X_L$, inductive reactance and $\frac{1}{C\omega}$=X_C, capacitive reactance

Therefore,

$$E_C = i_o \sqrt{R^2 + \left(X_L - X_C\right)^2}$$



If E makes the angle ϕ with ωt

$$\tan\phi = \frac{L\omega - \frac{1}{C\omega}}{R} = \frac{X_L - X_C}{R}$$

Then the potential difference E at any instant can be given by,

$$E = E_o \sin(\omega t + \phi)$$

The current, therefore **lags behind the potential** if $X_L > X_C$ and **leads** if $X_C > X_L$

Resonance

LCR series circuit

The impedance

$$Z = \sqrt{R^2 + \left(L\omega - \frac{1}{\omega C}\right)^2}$$

The impedance become lowest when $\left(L\omega - \frac{1}{\omega C}\right)$ is zero.

That is when $L\omega = \frac{1}{\omega C}$ or $X_L = X_C$

Then, $\omega^2 = \frac{1}{LC}$ $\qquad \omega = \frac{1}{\sqrt{LC}}$

Since $\tan\phi = \frac{X_L - X_C}{R}$, phase angle $\phi = 0$, when $X_L = X_C$

Then the current of this circuit is in phase with the voltage.

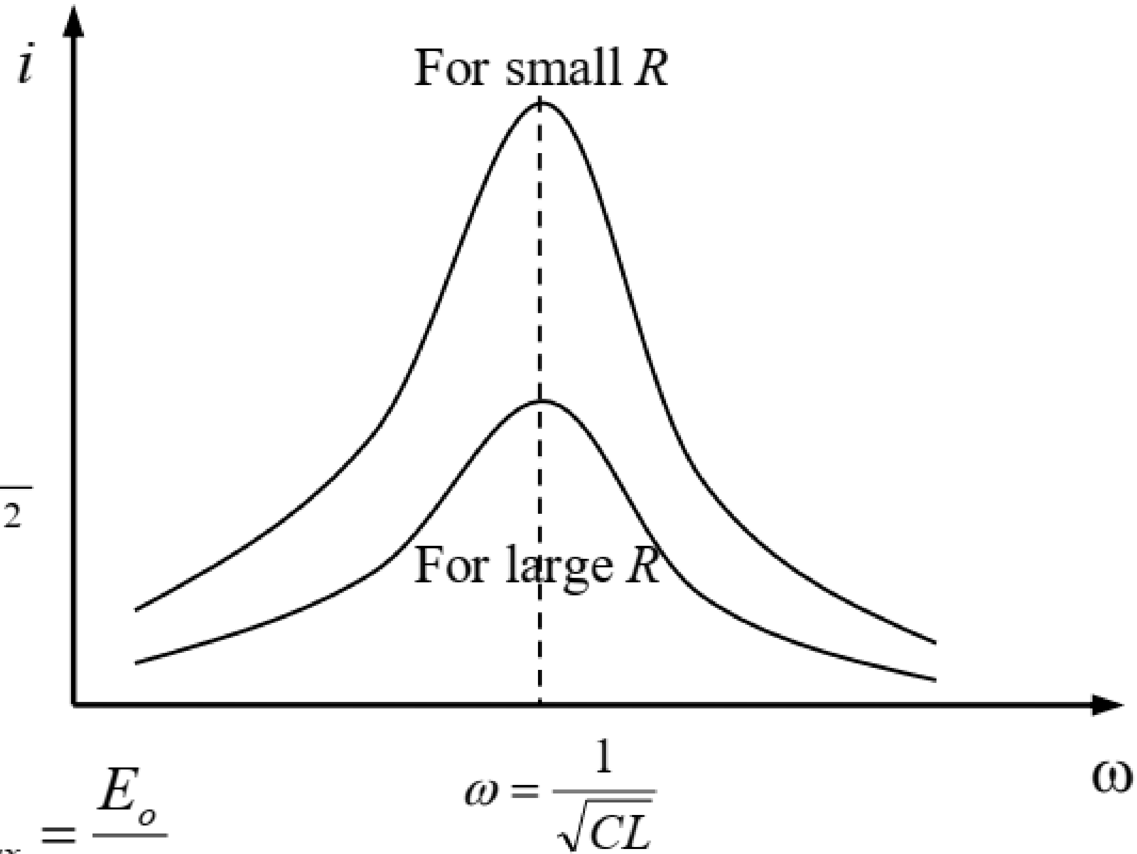
This condition, where voltage and the current are in phase is called the *resonance* condition.

The given LCR series circuit with constant values of inductance and capacitance can be in resonance only for a certain frequency *f* given by

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\frac{1}{\sqrt{LC}}$$

If the frequency of the impressed voltage equals this resonance frequency the current in the circuit reaches its highest.

$$E_o = i_o\sqrt{R^2 + \left(L\omega - \frac{1}{C\omega}\right)^2}$$



When $\omega = \frac{1}{\sqrt{CL}}$ $\quad i_{Max} = \frac{E_o}{R}$

Bandwidth and Quality Factor

For LCR series circuits we have shown that the voltage of the circuit is

$$E = i\sqrt{R^2 + \left(L\omega - \frac{1}{C\omega}\right)^2}$$

Therefore current of the circuit is

$$i = \frac{E}{\sqrt{R^2 + \left(L\omega - \frac{1}{C\omega}\right)^2}}$$

At resonance $i_0 = \dfrac{E}{R}$ since $\left(L\omega - \dfrac{1}{C\omega}\right) = 0$

Then above equation can be arranged as

$$i = \frac{E/R}{\sqrt{1 + \frac{1}{R^2}\left(L\omega - \frac{1}{C\omega}\right)^2}}$$

$$i = \frac{i_0}{\sqrt{1 + \frac{1}{R^2}\left(L\omega - \frac{1}{C\omega}\right)^2}}$$

$$\frac{i}{i_0} = \frac{1}{\sqrt{1 + \frac{\omega_0^2 L^2}{R^2}\left(\frac{L\omega}{\omega_0 L} - \frac{1}{\omega\omega_0 LC}\right)^2}}$$

Now we define the quality factor as $Q = \frac{\omega L}{R}$

Also we know that $\omega = \frac{1}{\sqrt{LC}}$

Substituting them in the above equation we get

$$\frac{i}{i_0} = \frac{1}{\sqrt{1 + Q^2\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)^2}}$$

Taking $\omega' = \frac{\omega}{\omega_0}$ we get

$$\frac{i}{i_0} = \frac{1}{\sqrt{1 + Q^2\left(\omega' - \frac{1}{\omega'}\right)^2}}$$

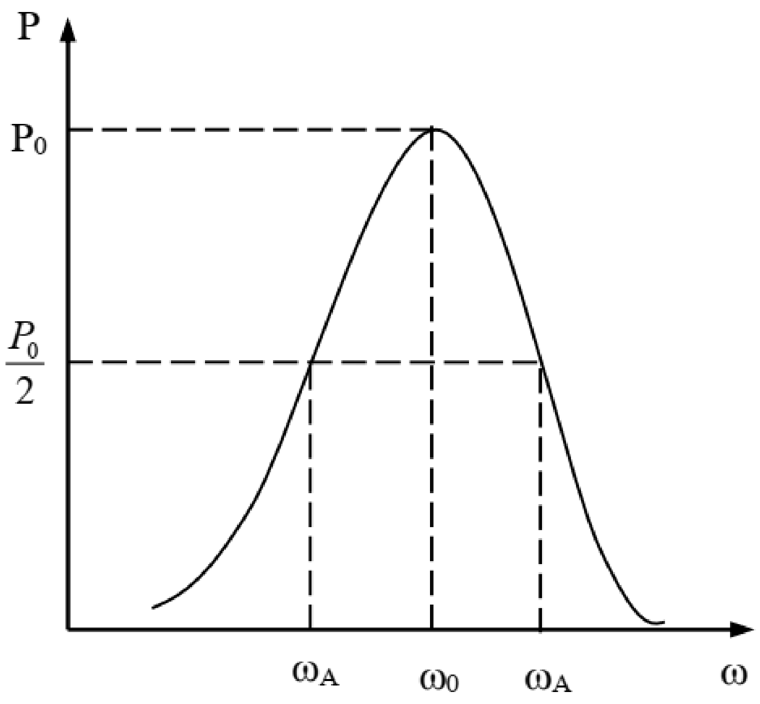
The power, $P \propto I^2$

Therefore

$$\frac{P}{P_0} = \frac{1}{1 + Q^2\left(\omega' - \frac{1}{\omega'}\right)^2}$$

The bandwidth is conventionally measured as the

difference of ω' at which the power consumption falls to half of the maximum value.



Therefore, When $P = \frac{P_0}{2}$

$$\frac{P_0/2}{P_0} = \frac{1}{1+Q^2\left(\omega' - \frac{1}{\omega'}\right)^2}$$

$$1+Q^2\left(\omega' - \frac{1}{\omega'}\right)^2 = 2$$

$$\left(\omega' - \frac{1}{\omega'}\right) = \pm\frac{1}{Q}$$

This represents quadratic equations

$$\omega'^2 - \frac{\omega'}{Q} - 1 = 0 \qquad \text{and} \qquad \omega'^2 + \frac{\omega'}{Q} - 1 = 0$$

The solutions of these two equations are

$$\omega'_A = \frac{1}{2Q} \pm \sqrt{\frac{1}{4Q^2} + 1} \qquad \text{and} \qquad \omega'_B = -\frac{1}{2Q} \pm \sqrt{\frac{1}{4Q^2} + 1}$$

Since ω' become negative when (-) sign in the ± mark is

considered, we take only the positive mark.

Then $\omega'_A = \frac{1}{2Q} + \sqrt{\frac{1}{4Q^2} + 1}$ and $\omega'_B = -\frac{1}{2Q} + \sqrt{\frac{1}{4Q^2} + 1}$

Therefore the difference $\omega'_A - \omega'_B = \frac{1}{Q}$

Since we have taken $\omega' = \frac{\omega}{\omega_0}$, $\frac{\omega_A - \omega_B}{\omega_0} = \frac{1}{Q}$

$$\boxed{Q = \frac{\omega_0}{\Delta\omega}}$$