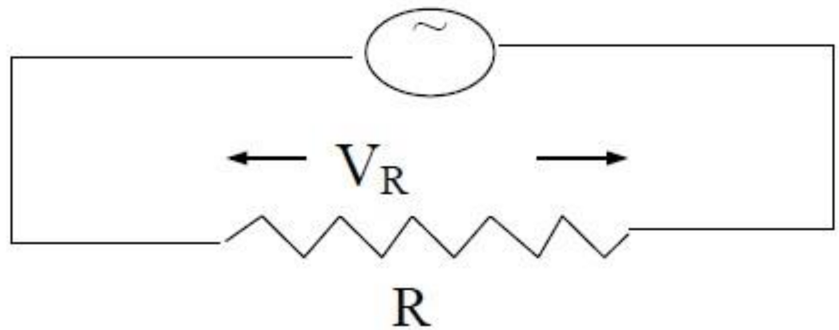
Alternating current Circuit Theory

AC Circuit with Resistor

consider a circuit containing a resistor R and source of alternating current (or voltage) as shown in the figure.



Let the current supplies by the A.C. source through the circuit be

$$i = i_o \sin \omega t$$

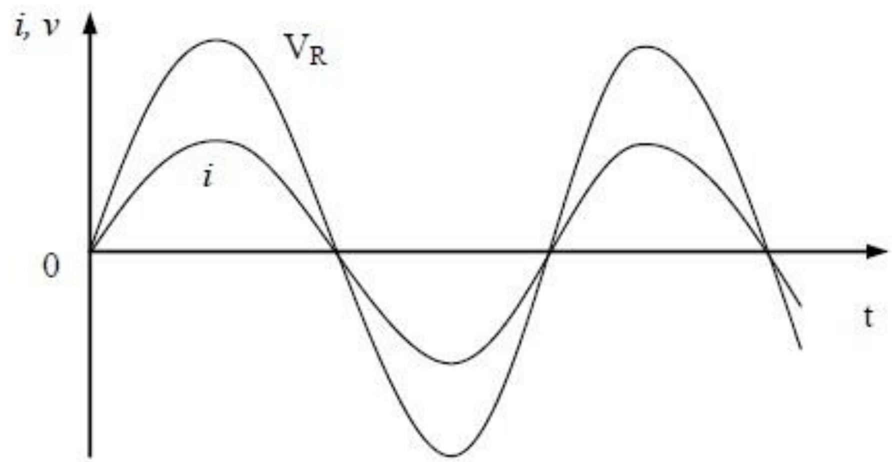
The instantaneous potential difference V_R across the resistor is

$$V_R = iR = i_o R \sin \omega t = V_{OR} \sin \omega t$$

Where $V_{OR} = i_o R$

That the potential difference across the resistor varies sinusoidally and also is in phase with the current.

That is both I and V_R reach their maximum values at the same time as shown in the figure



To find the rms values of the instantaneous voltage,

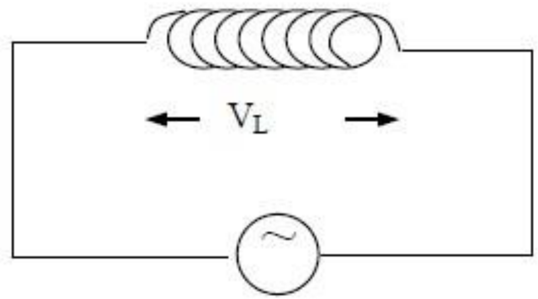
$$(V_R)_{rms} = R(i_o \sin \omega t)_{rms} = Ri_{rms}$$

Thus, Ohm's law is applicable to the A.C. circuit of the figure.

A.C. Circuit with an Inductor

An AC circuit containing pure inductor is shown in figure.

Even though an inductor always has some resistance, we assume that it is small and may be neglected.



The current through the circuit be $I_Z = I_o \sin \omega t$

The voltage drop V_L across the inductor is $V_L = L \frac{dI_Z}{dt}$, where L is the inductance.

$$V_L = L\frac{d}{dt}(I_o \sin\omega t) = LI_o \cos\omega t$$

When $\cos\omega t = 1$ (ie. $\omega t = 0$),

V_L takes its maximum value.

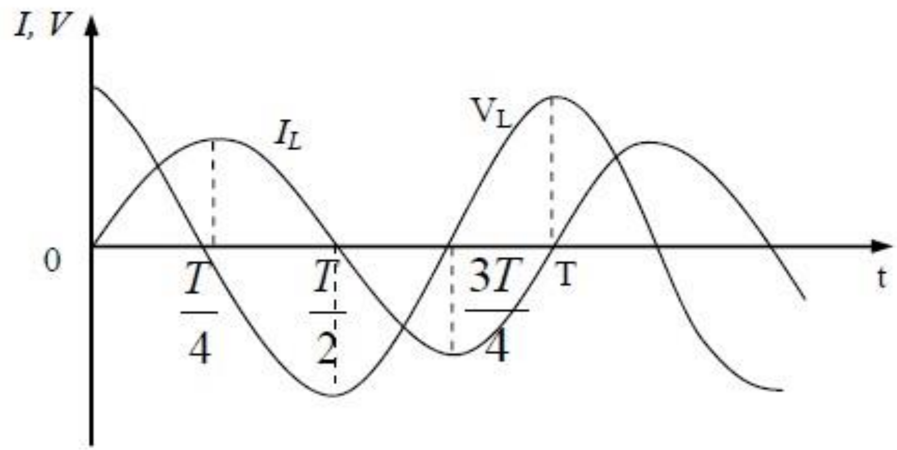
Let it be V_{0L}

Then $V_{L(\max)} = LI_o\omega t = V_{OL}$

Therefore $V_L = V_{OL}\cos\omega t$

$$\cos\omega t = \sin\left(\omega t + \frac{\pi}{2}\right) \qquad V_L = V_{0L}\sin\left(\omega t + \frac{\pi}{2}\right)$$

The plot of V_L and I_Z versus t is shown in figure.



This indicates that the voltage drop V_L across the inductor is not in phase with the current.

The voltage drop V_L across a pure inductor is $\frac{\pi}{2}$ or 90° or ¼ cycle ahead of the current I_Z.

This means that when the current I_Z is zero, V_L has its maximum value.

Graphs of V_L and I_Z versus t showing that V_L leads I_Z by $\frac{\pi}{2}$.

Using the equation $V_L = \omega L i_o \cos \omega t$

The instantaneous value of V_L can be taken.

$$\left(V_L\right)_{rms} = \omega t\left(i_o \cos \omega t\right)_{r.m.s}$$

$$\left(\mathrm{V_L}\right)_{\mathrm{rms}} = \omega \mathrm{L i_{rms}}$$

Comparing this equation with Ohm's law (V = IR).

The quantity ωL behaves like a resistance.

This is called the ***inductive reactance X_L.***

Then, $X_L = \omega L,$ where the units of X_L are also Ohms.

Then the above equation can be written as:

$$(V_L)_{r.m.s} = X_L i_{rms}$$

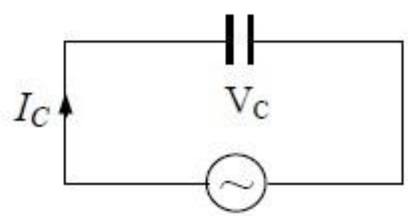
This is the form of Ohm's law for an inductor in an A.C circuit where X_L increases with increasing L.

AC Circuit with Capacitance

An A.C. circuit containing a capacitor is shown in the figure.

The current I_C through the circuit at any instant is given by

$$I_C = I_0 \sin \omega t$$



The current entering into the one plate of the capacitor must be equal to the current leaving the other plate.

So that the charges on the two plates are equal and opposite.

The instantaneous voltage V_C across the capacitor is,

$V_c = \frac{q}{C}$ Where q is the instantaneous charge on the capacitor.

To calculate V_C, q must be calculated from $I_C = I_0 \sin \omega t$

The relation between charge q and current I_C is

$$\frac{dq}{dt} = I_C = I_0 \sin \omega t$$

Then $dq = I_0 \sin \omega t \, dt$

Integrating $\int dq = \int I_0 \sin \omega t \, dt$

$$q = -\frac{I_0}{\omega} \cos \omega t$$

Substituting this value in the above equation $V_C = \frac{q}{C}$

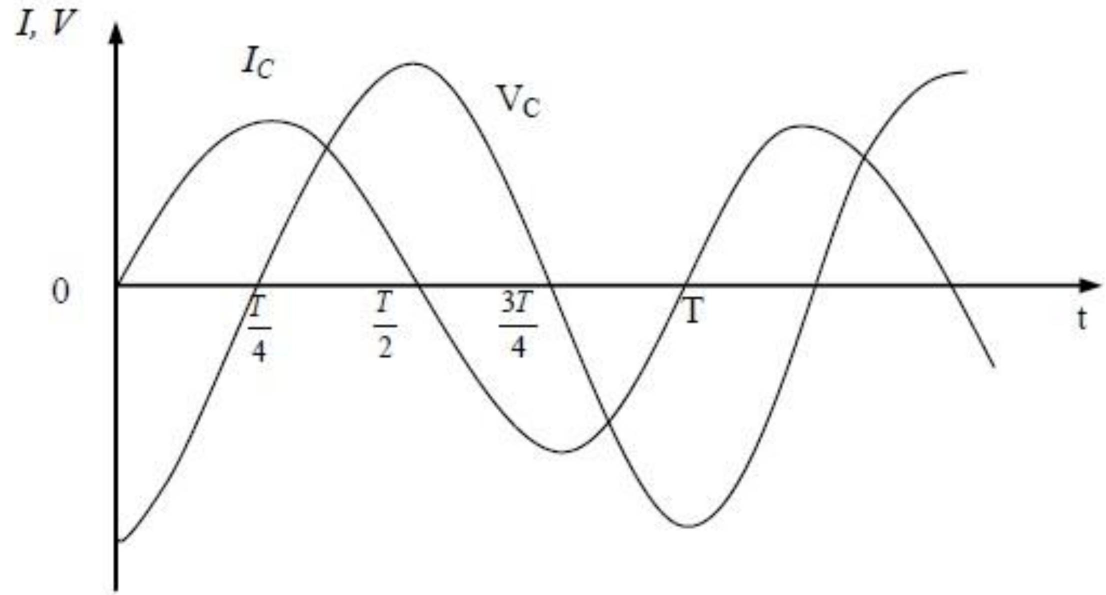
$$V_C = -\frac{I_o}{\omega C}\cos\omega t$$

$$V_C = -V_{oC}\cos\omega t$$

$$= V_o \sin\left(\omega t - \frac{\pi}{2}\right)$$

Where $V_{oC} = \left(\frac{I_o}{\omega C}\right) = I_o\left(\frac{1}{\omega C}\right)$

The plot of I_C and V_C versus t indicate that the V_C and I_C are not in phase (i.e. both i and V_C do not reach their maximum value at the same time).



The r.m.s. values

$$V_{C(r.m.s.)} = \frac{1}{\omega C}\left(-I_o \cos \omega t\right)_{r.m.s.}$$

$$V_{C(r.m.s.)} = \frac{1}{\omega C} I_{r.m.s}$$

On comparing this result with Ohm's law, the quantity $\frac{1}{\omega C}$ is seen to behave like a resistance.

It is called the capacitive reactance X_C.

$$X_C = \frac{1}{\omega C} \quad \text{where } \omega = 2\pi f$$

f-frequency of the A.C. and unit of X_C is ohms.

Unlike R, X_C is not constant, it decreases with increasing C, and ω.

Thus for A.C. current of large ω, ie. high frequency, the capacitor is an easy bypass. (i.e. the current goes through it easily) while for $\omega = 0$, it has infinite resistance.

Hence no D.C. passes through a capacitor.

Example:-

Calculate the resistance or the reactance and the voltage across each of the following for an A.C. source with an effective current of 5A and a frequency of 60 Hz, 600 Hz and 6000 Hz for

(a) resistor of 10Ω,

(b) Inductor of 1H, and

(c) Capacitor of 1μF. Make plots of R, X_L and X_C versus ω.

Solution

(a) The resistance R is independent of frequency and hence remains constant at 10Ω. Also $V_R = IR$ remains constant,

$$V_R = RI = 10\Omega \times 5A = 50A = 50V$$

(b) From equation $X_L = \omega L$

Where $\omega = 2\pi f$ and f is the frequency

Then $X_L = 2\pi f L$ and $V_L = IX_L = 2\pi fLI$

Therefore for, $f = 60s^{-1}$ $f = 600\,s^{-1}$ $f = 6000s^{-1}$

$$X_L = 377\Omega \quad X_L = 37770\Omega \quad V_L = 188500V$$

$$V_C = 1885V \quad V_L = 18850V \quad V_L = 188500V$$

(c) From the equation $X_C = \frac{1}{\omega C}$

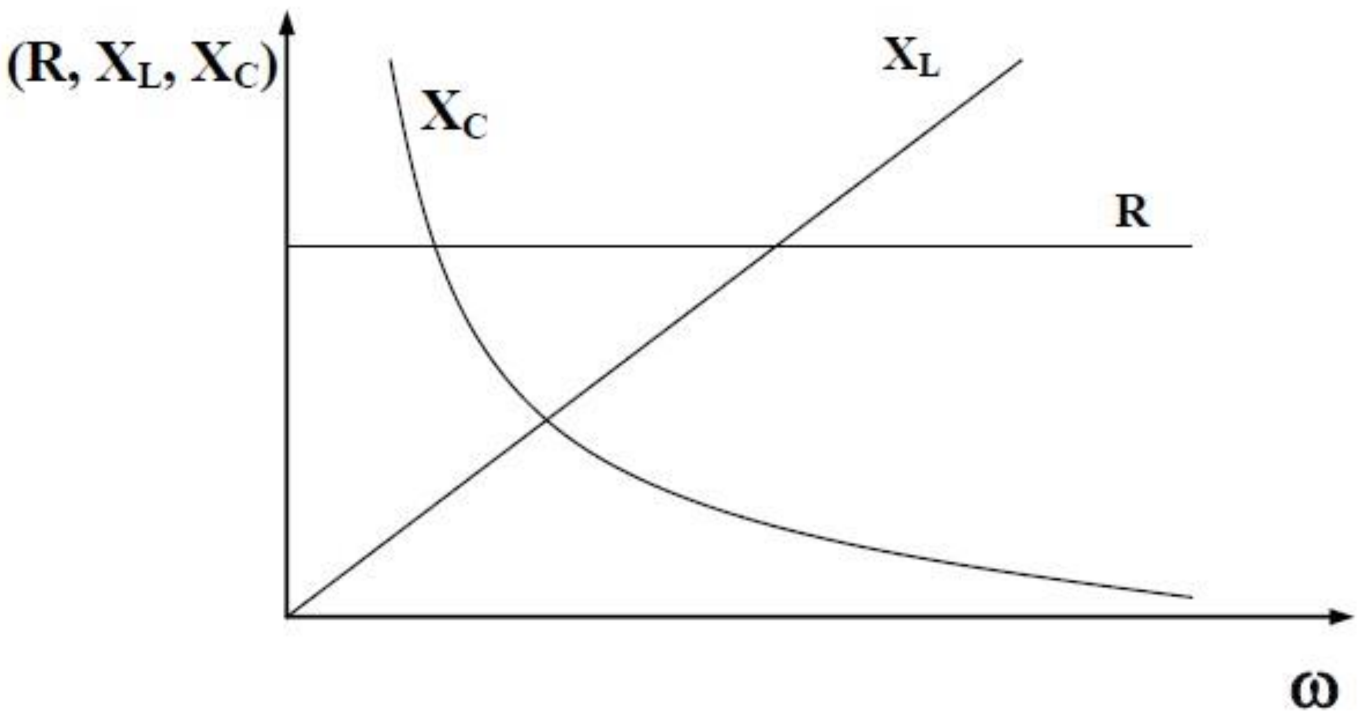
$$X_C = \frac{1}{2\pi \times 10^{-6}F} \cdot \frac{1}{f}$$

And from equation $V_C = X_C I = 5X_C$

Therefore, for $f = 60s^{-1}$ $f = 600\,s^{-1}$ $f = 6000s^{-1}$

$$X_C = 2650\Omega, \quad 265\Omega, \quad 26.5\Omega$$

$$V_C = 13250\,V, \quad 1325\,V, \quad 132.5\,V$$

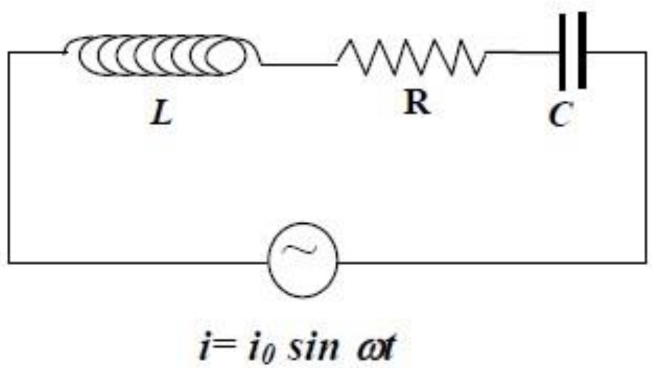


The plots of R, X_L and X_C. versus ω are shown in the figure.

Note that X_L increases with ω , X_C decreases, R remains constant.

The L.R.C Series Circuit

A more general case of an AC circuit is the one that contains all three elements, resistance inductance and capacitance as shown in the figure.



Let the instantaneous current i through any point in the circuit be given by $I = I_0 \sin \omega t$

V_R, V_L and V_C are the instantaneous voltages across the resistance, The inductance and the capacitance respectively.

Then, we may write $$V = V_R + V_L + V_C$$

Which states that the instantaneous voltage V across the generator is equal to the sum of the voltages across the elements.

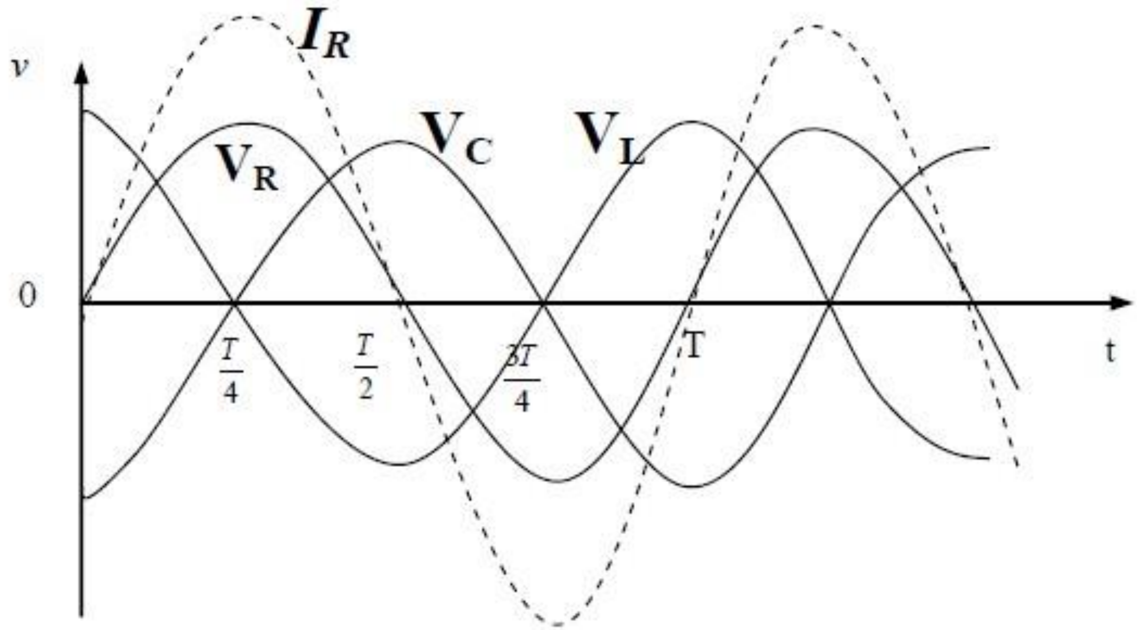
Then $V_R = IR = I_0 R \sin \omega t$

$$V_L = L\frac{dI}{dt} = I_0 L\omega \cos \omega t = I_0 X_L \cos \omega t$$

where $X_L = \omega L$

$V_C = -I_0 X_C \cos \omega t$; where $X_c = \frac{I}{\omega C}$

The plots of V_L, V_R, V_C, with i are shown in the figure



Substituting these values for the voltages.

$$V = I_0\left[R\sin\omega t + \left(X_L - X_C\right)\cos\omega t\right]$$

After multiplying and dividing by, $\sqrt{R^2 + (X_L - X_C)^2}$

$$V = I_0\sqrt{R^2 + (X_L - X_C)^2}\left(\frac{R}{\sqrt{R^2 + (X_L - X_C)^2}}\sin\omega t + \frac{X_L - X_C}{\sqrt{R^2 + (X_L - X_C)^2}}\cos\omega t\right)$$

Let $\cos\phi = \dfrac{R}{\sqrt{R^2 + (X_L - X_C)^2}}$ and $\sin\phi = \dfrac{X_L - X_C}{\sqrt{R^2 + (X_L - X_C)^2}}$

Then the above equation takes the form

$$V = I_0\sqrt{R^2 + (X_L - X_C)^2}\{\cos\phi\sin\omega t + \sin\phi\cos\omega t\}$$

or $$V = I_0\sqrt{R^2 + (X_L - X_C)^2}\{\sin(\omega t + \phi)\}$$

Where, $$\tan\phi = \frac{X_L - X_C}{R}$$

Then equation can be written in the form

$$V = V_o \sin(\omega t + \phi)$$

Where $$V_o = I_0\sqrt{R^2 + (X_L - X_C)^2}$$

If $\sqrt{R^2 + (X_L - X_C)^2} = Z$

$$\boxed{V_0 = I_o Z}$$

with $X_L = \omega L$ and $X_C = \frac{1}{\omega C}$

In this general case Z has taken the place of resistance and it is called the *impedance* of the circuit.

Comparison of equation $I = I_0 \sin \omega t$ and $V = V_0 \sin(\omega t + \phi)$ indicates that the voltage V and current i are out of phase by an angle ϕ.

The value of ϕ depends on the values of X_L, X_C and R.

If $X_L > X_C$, $\tan\phi = \frac{(X_L - X_C)}{R}$ is positive and hence ϕ is positive. **Then the voltage leads the current.**

If $X_C > X_L$, $\tan\phi = \frac{(X_L - X_C)}{R}$ is negative and hence ϕ is negative. **Then the current leads the voltage.**

Impedance

The joint effect of resistance and reactance in an AC circuit is known as **impedance**. It is designated by the symbol **Z.**

Impedance is defined as the ratio of the effective voltage to the effective current.

The defining equation is

$$Z = \frac{E_{rms}}{I_{rms}}$$

Impedance is measured in Ohms since E_{rms} is in Volts and I_{rms} in Amperes.

(a) For a circuit containing only resistance and inductance, it is clear that,

$$E_o = i_o\left(R^2 + L^2\omega^2\right)^{1/2} = \sqrt{R^2 + X_L{}^2 i_o}$$

Therefore

$$Z = \sqrt{R^2 + X_L{}^2}$$

Then the current through the circuit is given by,

$$i_o = \frac{E_o}{\sqrt{R^2 + L^2\omega^2}}$$

(b) For a L-C-R series circuit, as shown above,

$$E_o = i_o\sqrt{R^2 + \left(L\omega - \frac{1}{C\omega}\right)^2} = i_0\sqrt{R^2 + (X_L - X_C)^2}$$

Therefore,

$$Z = \sqrt{R^2 + \left(X_L - X_C\right)}$$

Where X_L and X_C are the reactances of L and C respectively.

If X is the net reactance of the circuit $\quad X = X_L - X_C$

Then the impedance $Z = \sqrt{R^2 + X^2}$

Susceptance

The reciprocal value of the reactance is called the ***susceptance*** and it is denoted by the symbol **B.**

Therefore $B = \dfrac{1}{X}$

The unit of susceptance is **Ohm^{-1} (Ω^{-1})**

Admittance

This is defined as the reciprocal of ***impedance***. The symbol for admittance is **y.**

Thus $y = \frac{1}{Z}$

The unit of admittance is **Ohm^{-1} (Ω^{-1})**

Example:-

A resistor R, an inductance L and a capacitor C are all connected in series with an A.C. supply. The resistance of R is 15Ω and for the given frequency the inductive reactance of L is 24Ω and capacitive reactance of C is 14Ω. If the current in the circuit is 5A find,

(a) The potential difference across each of the components R, L and C.

Solution

$$R = 15\Omega, \quad \omega L = 24\Omega. \frac{1}{\omega C} = 14\Omega, i_o = 5A$$

(i) The potential difference across R is $V_R = i_o R$

$$V_R = 5 \times 15 = 75V$$

The potential difference across L is $V_L = i_o L\omega$

$$V_L = i_o L\omega = 5 \times 24 = 120V$$

The potential difference across C is $V_C = \frac{i_o}{C\omega}$

$$V_C = \frac{i_o}{C\omega} = 5 \times 14 = 70V$$

(ii) The impedance of the circuit is

$$Z = \left[R^2 + \left(L\omega - \frac{1}{\omega C} \right)^2 \right]^{1/2}$$

$$= \left[15^2 + (24 - 14)^2 \right]^{1/2}$$

$$= [225 + 100]^{1/2}$$

$$Z = 18.02\Omega$$

(iii) The voltage of the A.C. supply is obtained as

$$V_o = i_o Z$$

$$= 5 \times 325^{1/2} = 5 \times 18.02$$

$$\underline{\underline{V = 90.138V}}$$

(iv) The phase angle θ is given by

$$\tan\theta = \frac{X_L - X_C}{R} = \frac{L\omega - 1/\omega C}{R}$$

$$\tan\theta = \frac{24 - 14}{15} = \frac{19}{15}$$

$$\underline{\underline{\theta = 33°41'}}$$