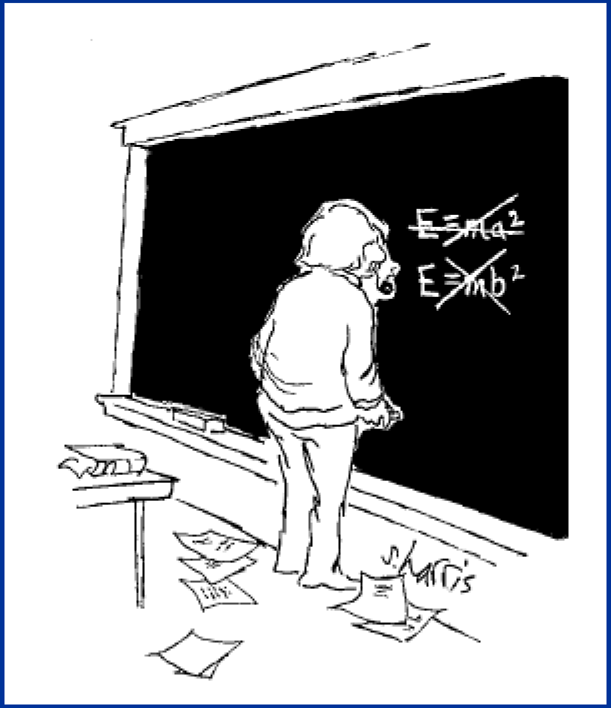
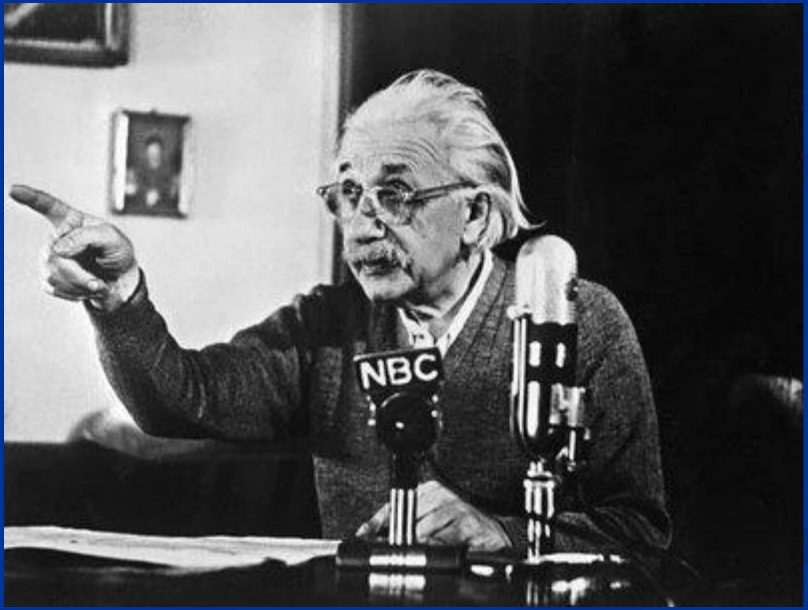
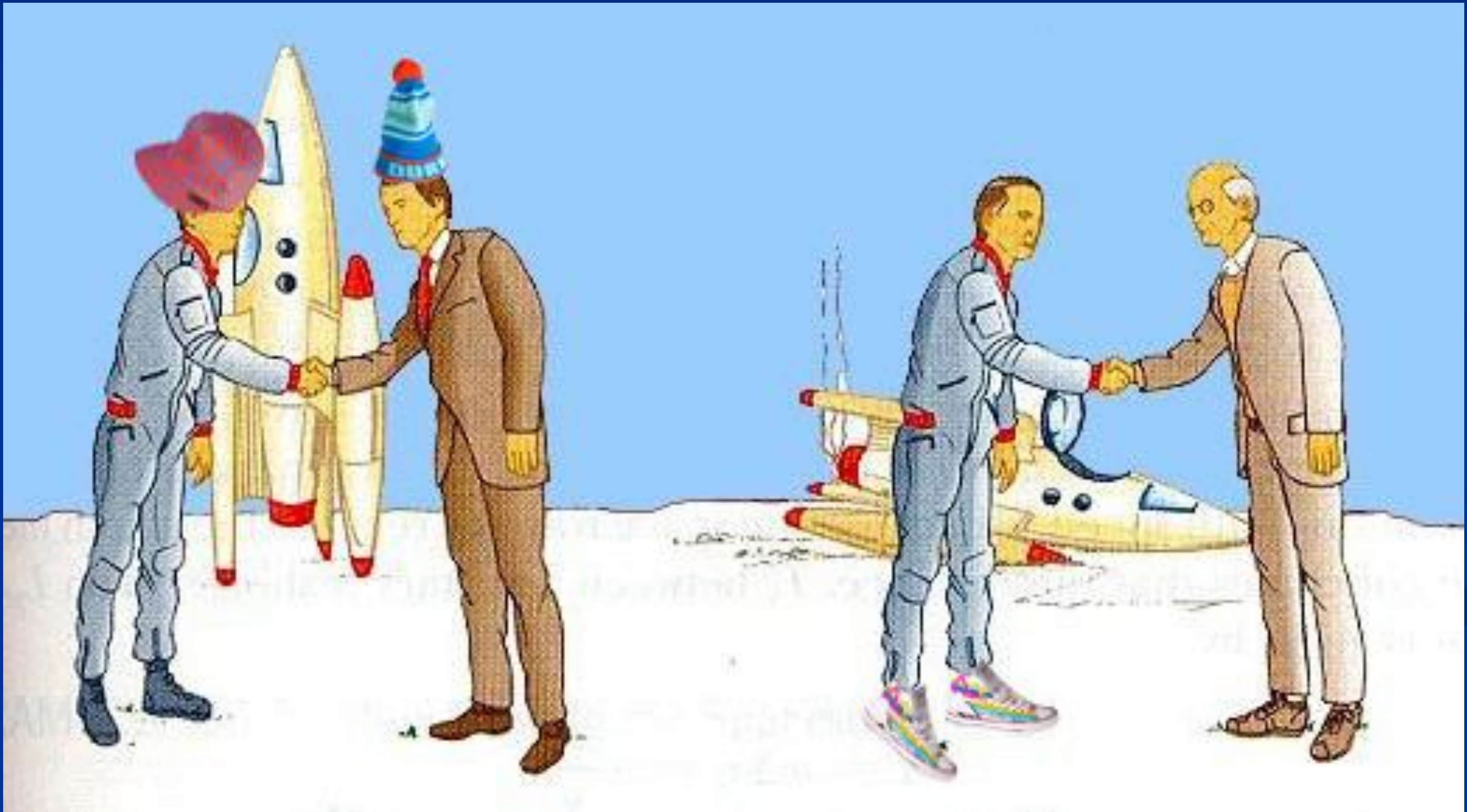
Special Theory of Relativity



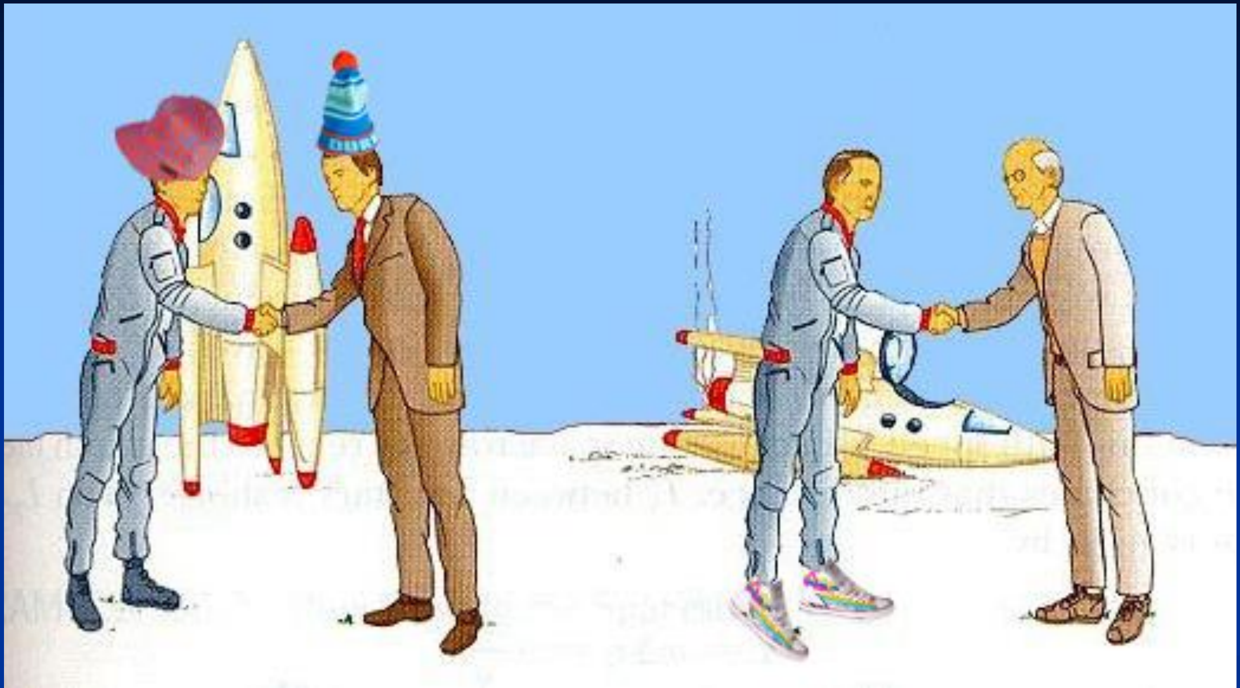


8th Lecture

Twin Paradox



Twin Paradox



Paradox :

A paradox is a statement or group of statements that leads to a contradiction or a situation which defines intuition.

The term is also used for an apparent contradiction that actually expresses a non-dual truth!

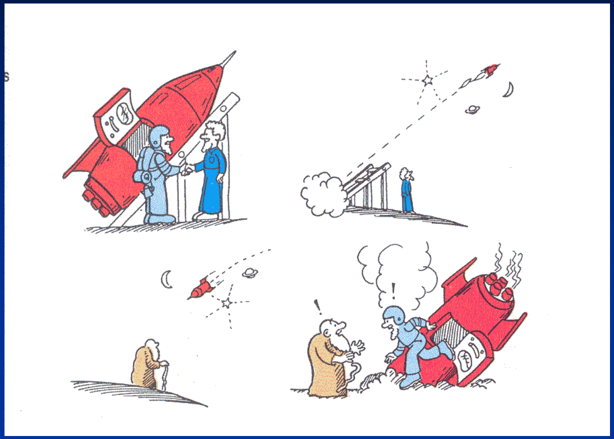
A statement or proposition seeming self-contradictory or absurd but in reality expressing a possible truth!

Twin Paradox

In Physics, the twin paradox is a thought experiment in special relativity, in which a twin makes a journey into space in a high speed rocket and returns home to find he has aged less than his identical twin who stayed on Earth.

This result appears puzzling because each twin sees the other twin as travelling and so, according to the theory of special relativity, paradoxically each should find the other to have aged more slowly!

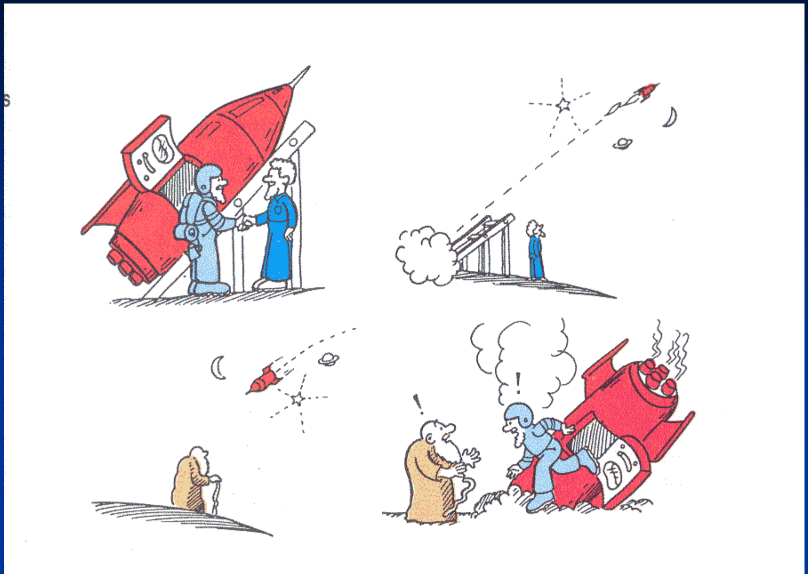
Twin Paradox



However, this scenario can be resolved within the standard framework of special relativity (because the twins are not equivalent; the space twin experienced additional, asymmetrical acceleration when switching direction to return home), and therefore is not a paradox in the sense of a logical contradiction.

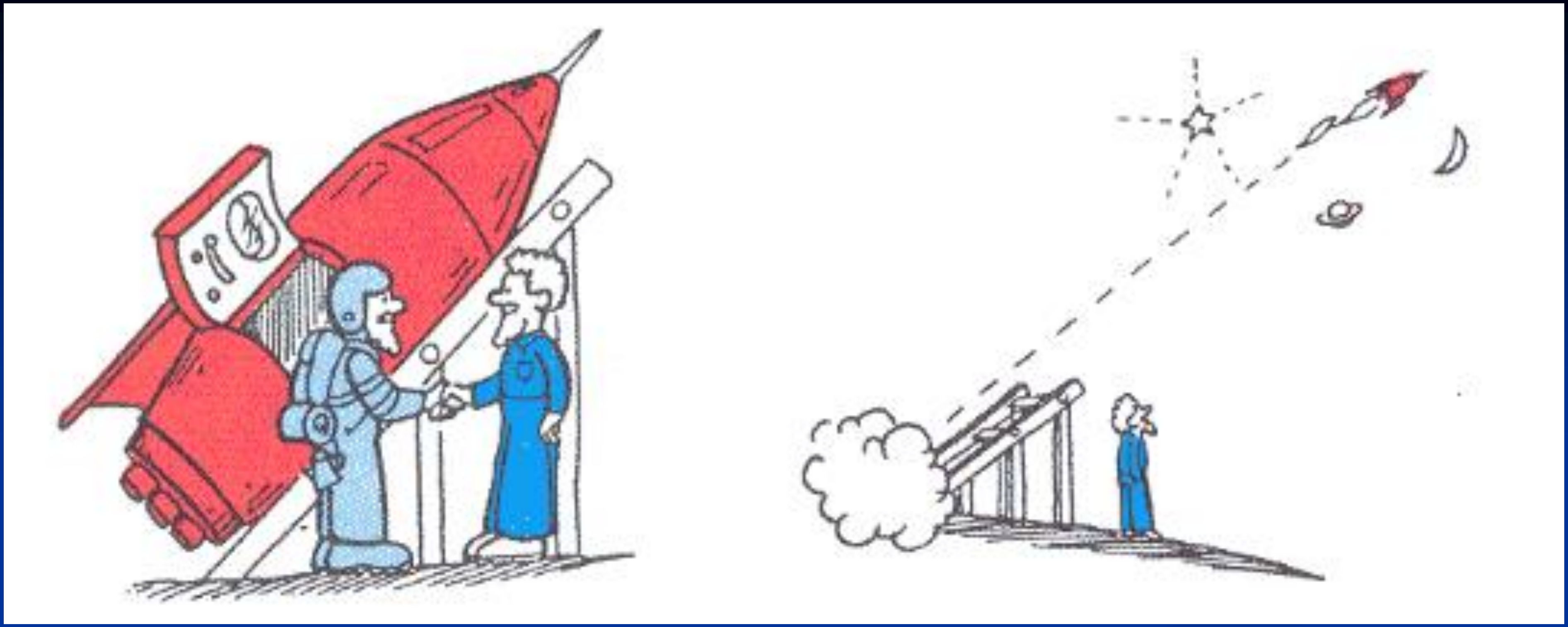
Starting with Paul Langevin in 1911, there have been numerous explanations of this paradox, many based upon there being no contradiction because there is no symmetry—only one twin has undergone acceleration and deceleration, thus differentiating the two cases.

Twin Paradox

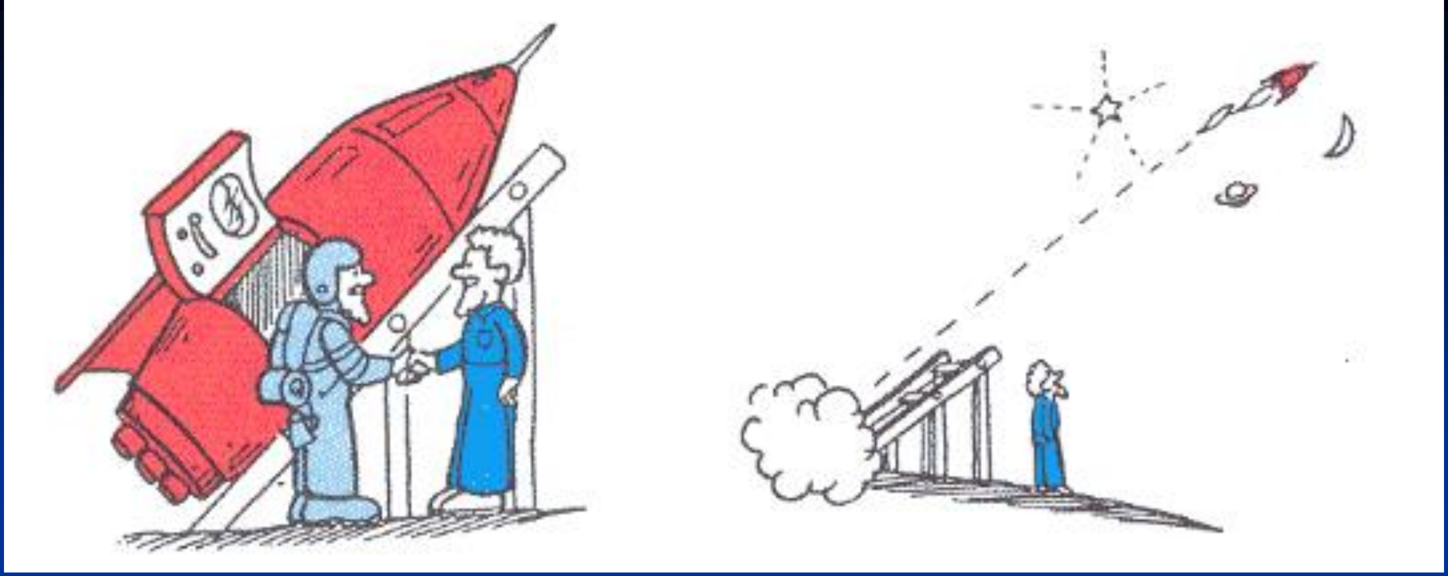


Max von Laue argued in 1913 that since the traveling twin must be in two separate inertial frames, one on the way out and another on the way back, this frame switch is the reason for the aging difference, not the acceleration *per se*.

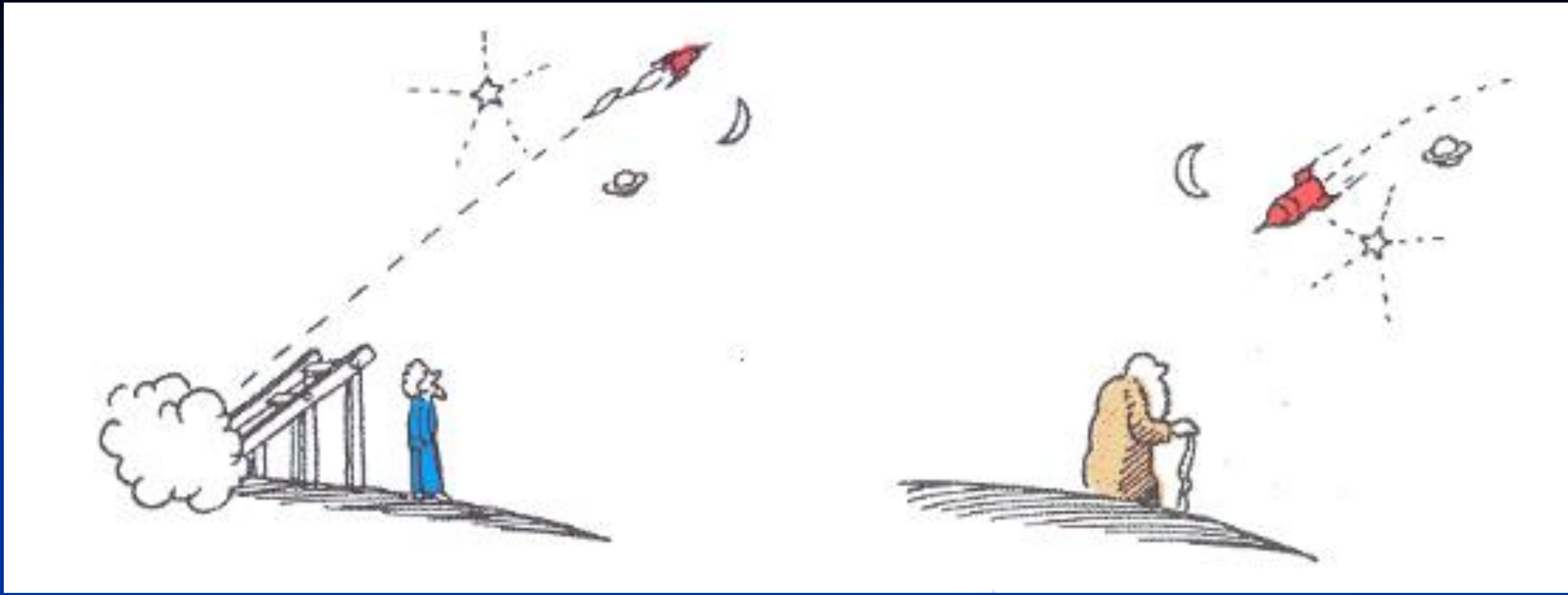
Explanations put forth by Albert Einstein and Max Born invoked gravitational time dilation to explain the aging as a direct effect of acceleration.



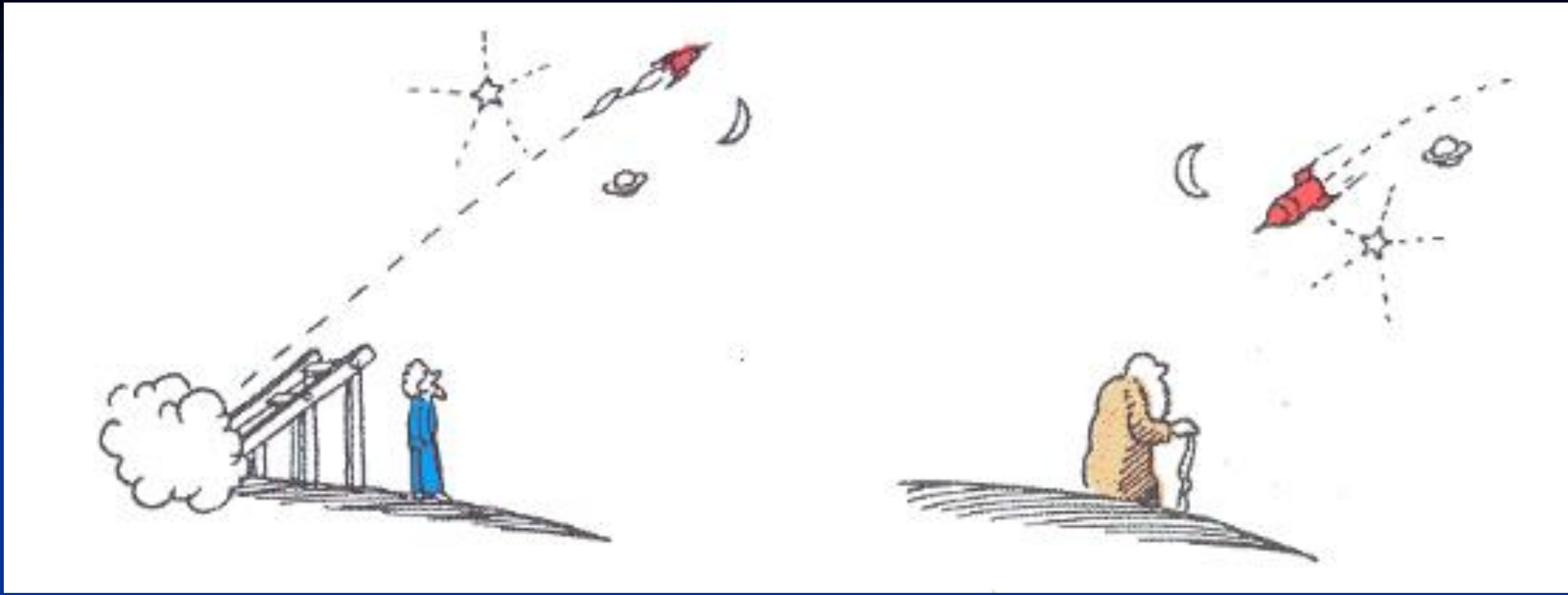
Consider a space ship traveling from Earth to the nearest star system outside of our solar system: a distance $d = 4$ light years away, at a speed $v = 0.8c$.



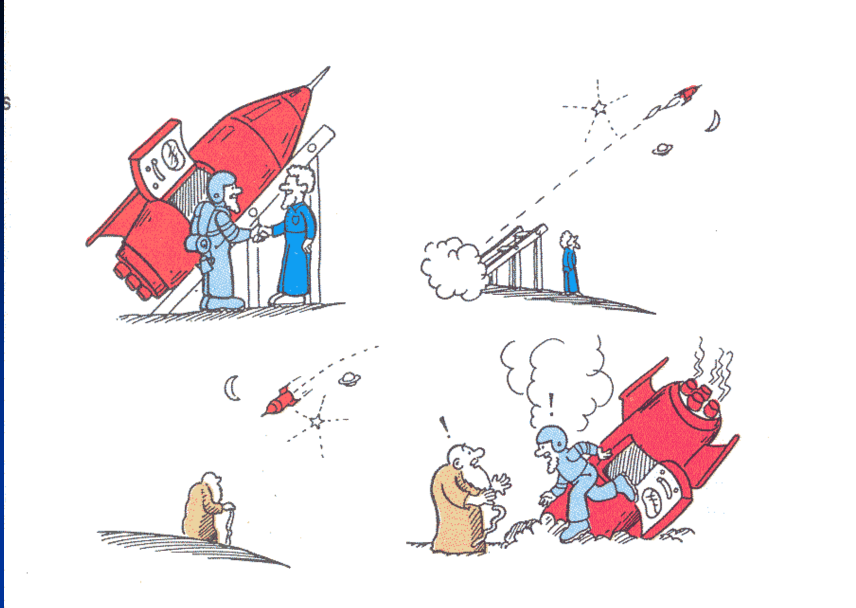
The Earth-based mission control reasons about the journey this way: the round trip will take $t = 2d/v = 10$ years in Earth time (*i.e.* everybody on Earth will be 10 years older when the ship returns). The amount of time as measured on the ship's clocks and the aging of the travelers during their trip will be reduced by the factor $\varepsilon = (1 - v^2/c^2)^{1/2}$. In this case $\varepsilon = 0.6$ and the travelers will have aged only $0.6 \times 10 = 6$ years when they return.



The ship's crew members also calculate the particulars of their trip from their perspective. They know that the distant star system and the Earth are moving relative to the ship at speed v during the trip. In their rest frame the distance between the Earth and the star system is $\varepsilon\, d = 0.6\, d = 2.4$ light years (length contraction), for both the outward and return journeys.



Each half of the journey takes $2.4/v = 3$ years, and the round trip takes $2 \times 3 = 6$ years. Their calculations show that they will arrive home having aged 6 years. The travelers' final calculation is in complete agreement with the calculations of those on Earth, though they experience the trip quite differently from those who stay at home.



If twins are born on the day the ship leaves, and one goes on the journey while the other stays on Earth, they will meet again when the traveler is 6 years old and the stay-at-home twin is 10 years old. The calculation illustrates the usage of the phenomenon of length contraction and the experimentally verified phenomenon of time dilation to describe and calculate consequences and predictions of Einstein's special theory of relativity.

Special Theory of Relativity

Einstein's Two Postulates in STR

Postulate 01 : The Principle of Relativity:

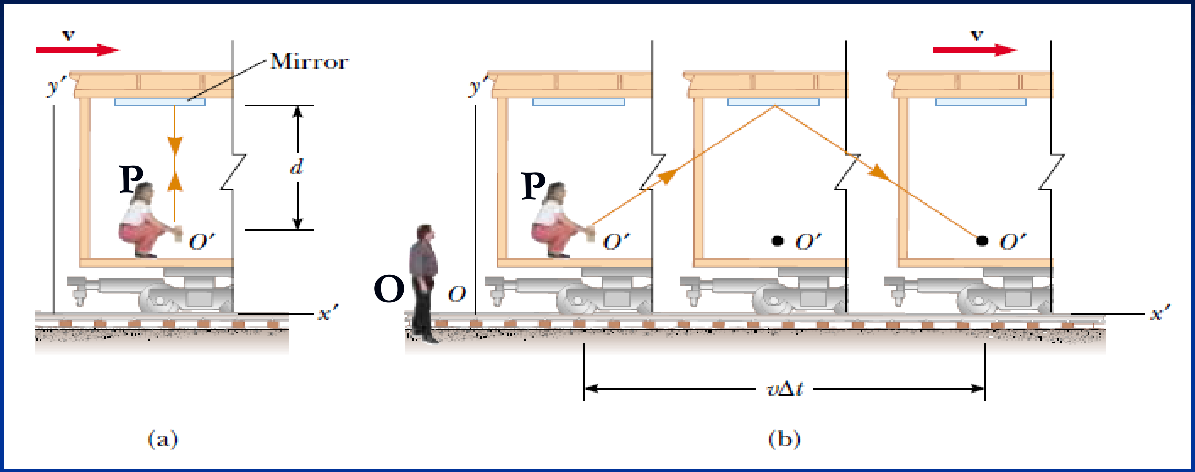
The laws of physics must be the same in all inertial reference frames.

The laws of Physics are the same for all observers in uniform motion relative to one another.

Postulate 02 : The constancy of the speed of light :

The speed of light in vacuum has the same value, $c = 3 \times 10^8$ m/s in all inertial frames, regardless of the velocity of the observer or the velocity of the source emitting the light.

Measurement of Time in STR



$$\Delta t_O = \Delta t_P \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

Where, v is the Relative Speed of the Two Frames

This equation express the fact that for the moving observer the period of the clock is shorter than in the frame of the ground observer itself !

$$\Delta t_O \rangle \Delta t_P$$

Time interval w. r. t the stationary frame

Time interval w. r. t the moving frame

Time Dilation

Suppose,

$$t_2 = t_1 \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

This equation is called relativistic time equation!

If $v > 0$ ➡ $\frac{v}{c} < 1$ ➡ $\frac{v^2}{c^2} < 1$ ➡ $1-\frac{v^2}{c^2} < 1$

➡ $\sqrt{1-\frac{v^2}{c^2}} < 1$ ➡ $t_2 = t_1 \frac{1}{(<1)}$

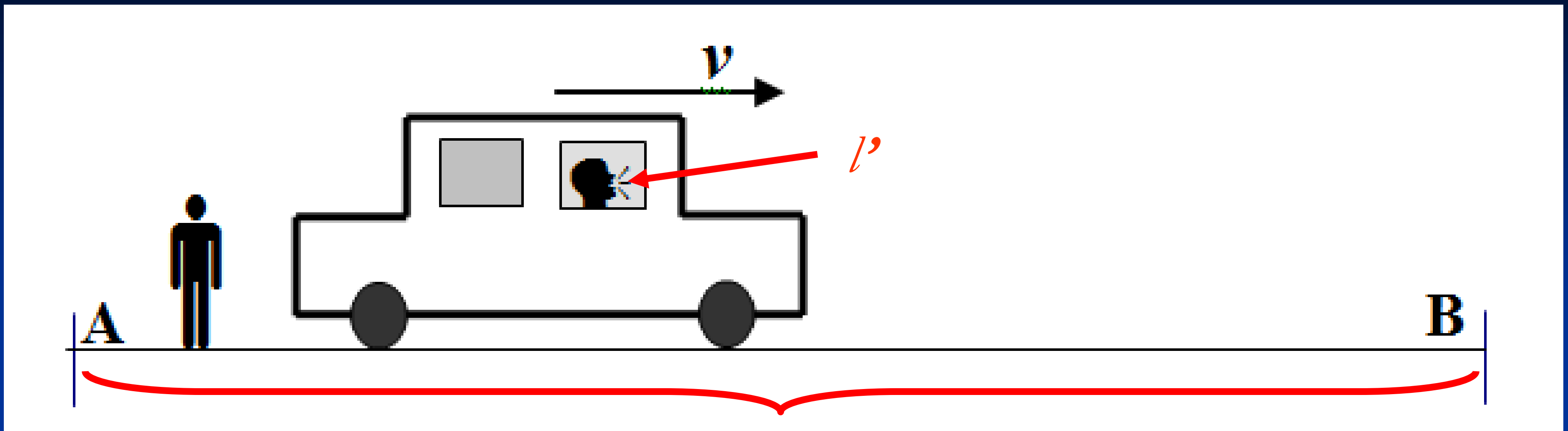
$$t_2 > t_1$$

t_2: Time interval w. r. t the stationary frame

t_1: Time interval w. r. t the moving frame

This is called Time Dilation !

Measurement of Length in STR



$$t_{im} = \frac{l}{v}$$

$$t_{pro} = \frac{l^1}{v}$$

$$t_{im} = t_{pro}\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

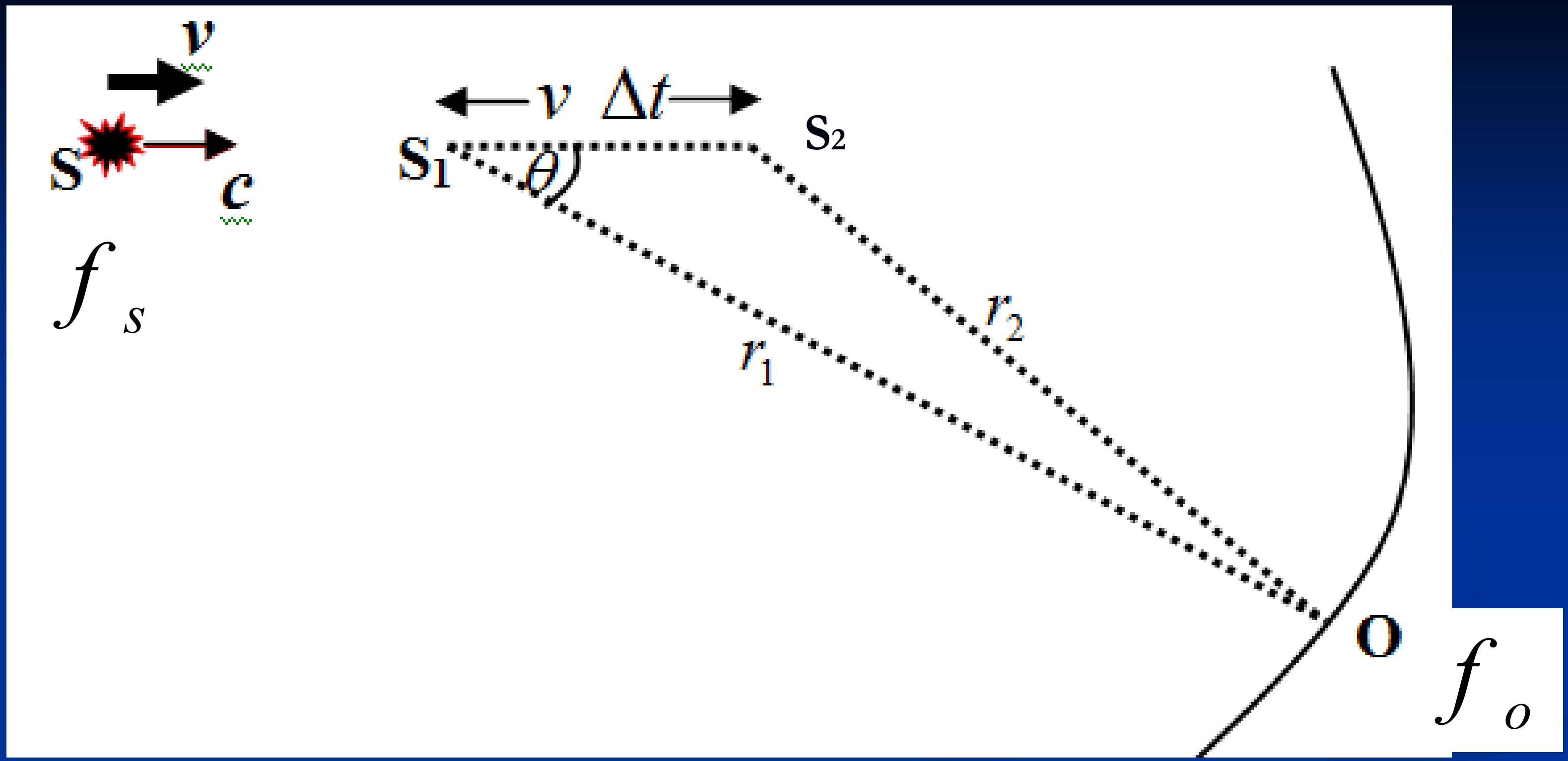
$$\Rightarrow \quad l^1 = l\sqrt{1-\frac{v^2}{c^2}}$$

If $v > 0$ $\Rightarrow$ $l^1 < l$

l^1: Length measured by an observer in the car

l: Length measured by observer in the Earth

This is called Length Contraction !



$$f_o = f_s \frac{1}{\gamma(1-\beta\cos\theta)}$$

where, $\gamma = \dfrac{1}{\sqrt{1-\beta^2}}$

and $\beta = \dfrac{v}{c}$

This is the general form of the Doppler's Effect in STR!

Relativistic Mass

If we assume a body with mass m is moving with a constant velocity, v.

Using the relationship between Relativistic Energy - Momentum:

$$E^2 = p^2c^2 + m_o^2c^4$$

Where, $E = mc^2$ and, $p = mv$

$$(mc^2)^2 = (mv)^2c^2 + m_o^2c^4$$

$$m = \frac{m_o}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Experimental Proofs :

The first verification of the increase in mass with velocity came from the experimental work of Kaufmann in 1902 and 1906 and particularly, that of Bucherer in 1909.

Sommerfeld's theory of Atomic Orbits:

This verification of the mass increased predicted by the STR was proposed by Sommerfeld in 1916.

Atomic Accelerators :

Early in 1952 the **Brookhaven National Laboratory** announced its success in accelerating protons (nuclei of H atoms) up to 0.95 c. As a result the mass of the proton was increased to about **three times** its original mass.

And in June 1952 the **California Institute of Technology** announced it had succeeded in accelerating electrons to 0.9999999 c. The corresponding mass increase was about **900 times** its original mass.

Twin Paradox

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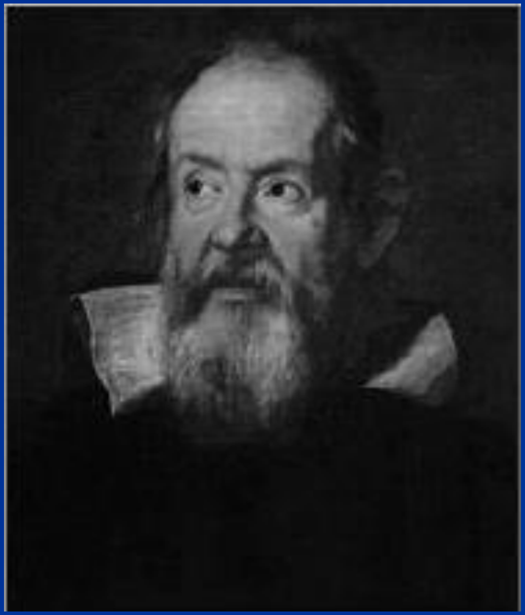
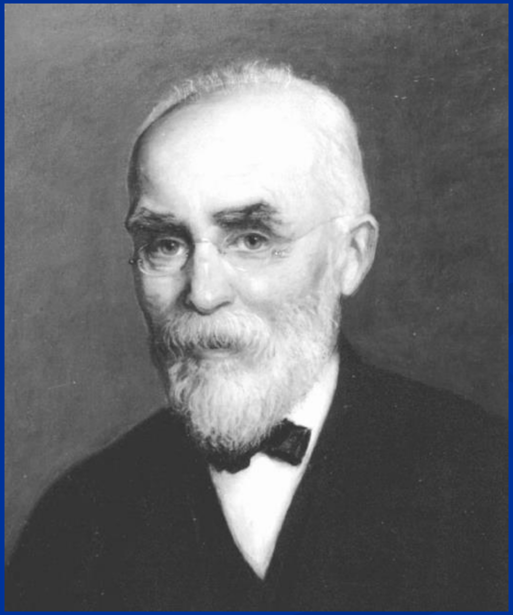
Transformations Equation



Galileo Galilei



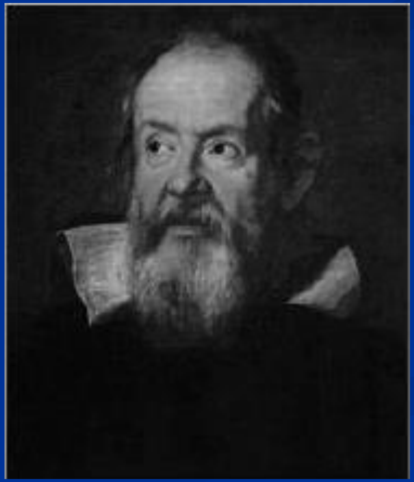
Hendric Lorentz

Transformations Equation

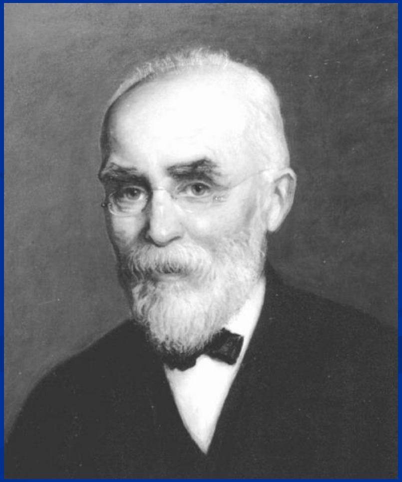
Transformation equations are used to transform between the coordinates of two reference frames. There are two types of transformation equations.

1. Galilean Transformation Equation (without relativistic effect!)

2. Lorentz Transformation Equation (with relativistic effect!)

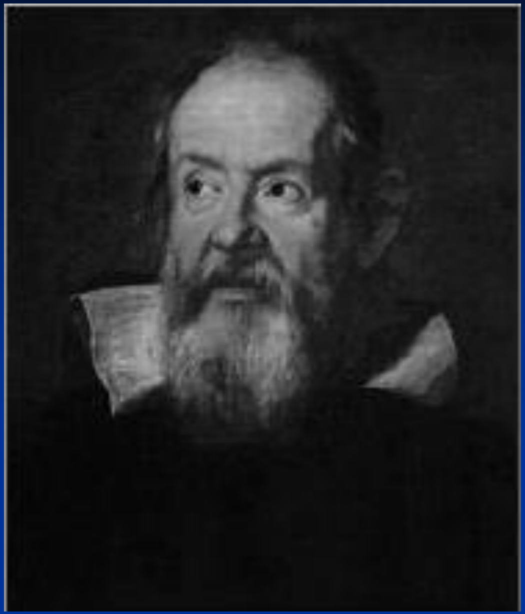


Galileo Galilei



Hendric Lorentz

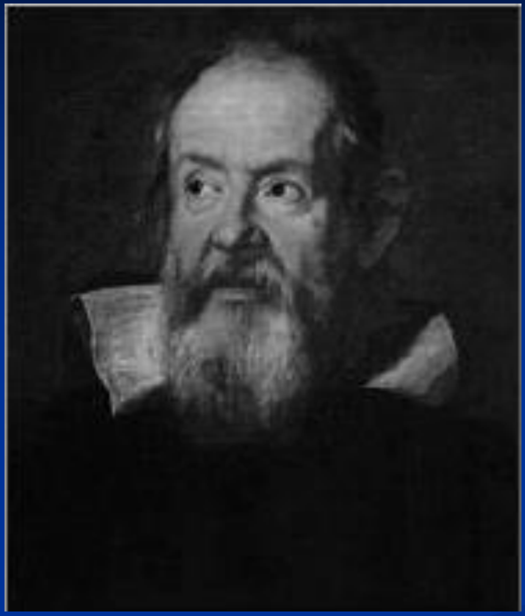
Galilean Transformation



The **Galilean transformation** is used to transform between the coordinates of two reference frames which differ only by constant relative motion within the constructs of Newtonian physics. This is the passive transformation point of view. The equations below, although apparently obvious, break down at speeds that approach the speed of light owing to physics described by relativity theory.

P. T. O

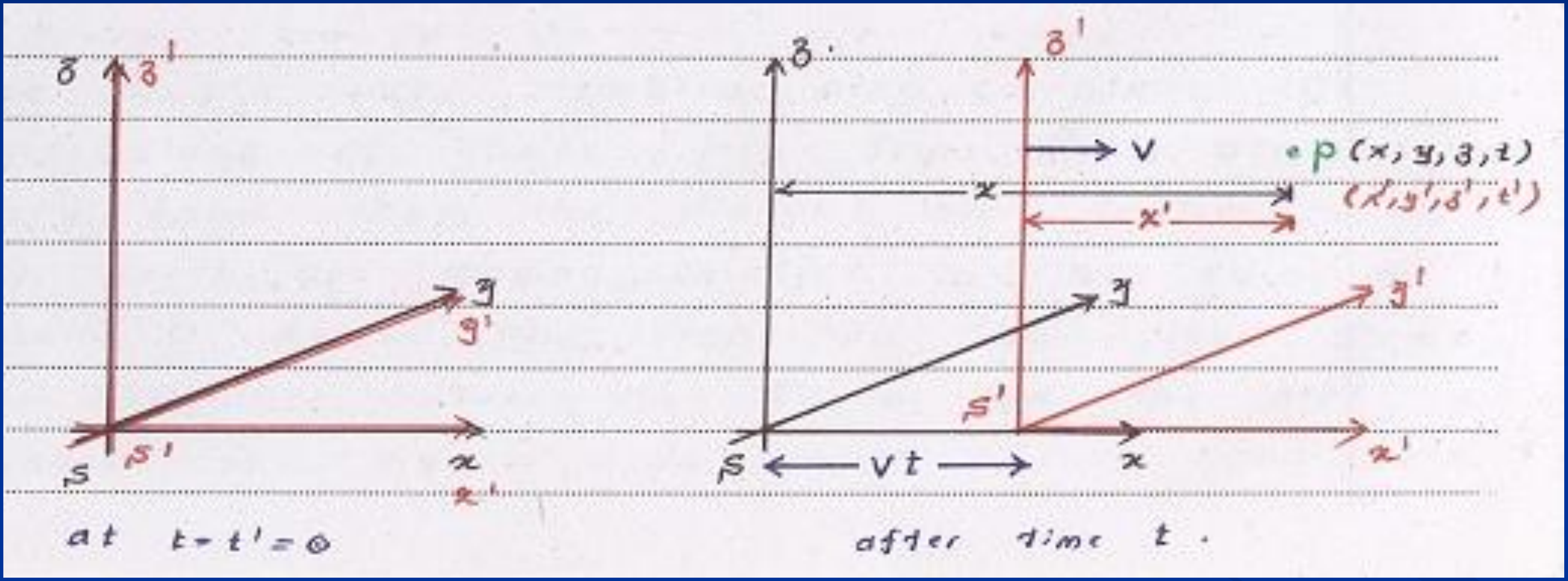
Galilean Transformation



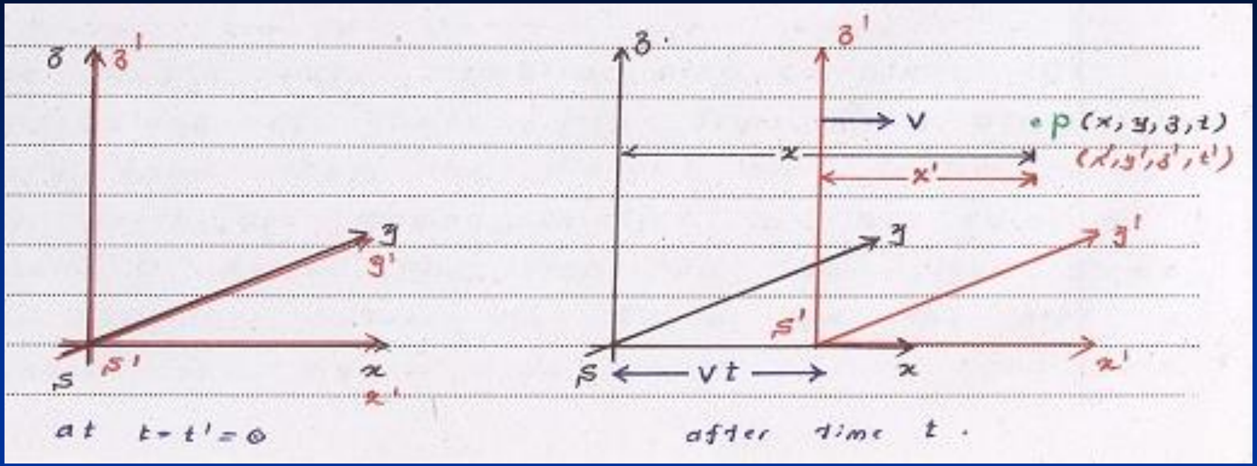
Galileo formulated these concepts in his description of *uniform motion*. The topic was motivated by Galileo's description of the motion of a ball rolling down a ramp, by which he measured the numerical value for the acceleration of gravity near the surface of the Earth.

P. T. O

Galilean Transformation



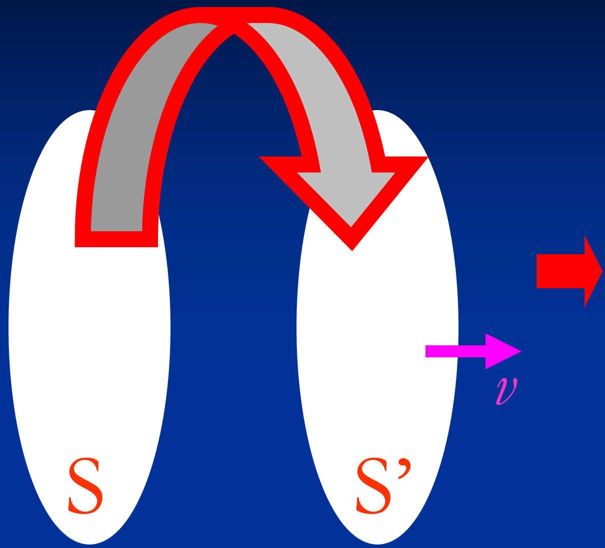
Galilean Transformation



The notation below describes the relationship under the Galilean transformation between the coordinates (x, y, z, t) and (x', y', z', t') of a single arbitrary event, as measured in two coordinate systems S and S', in uniform relative motion (velocity v) in their common x and x' directions, with their spatial origins coinciding at time $t = t' = 0$:

$$
\begin{aligned}
x' &= x - vt \\
y' &= y \\
z' &= z \\
t' &= t
\end{aligned}
$$

Galilean Inverse Transformation

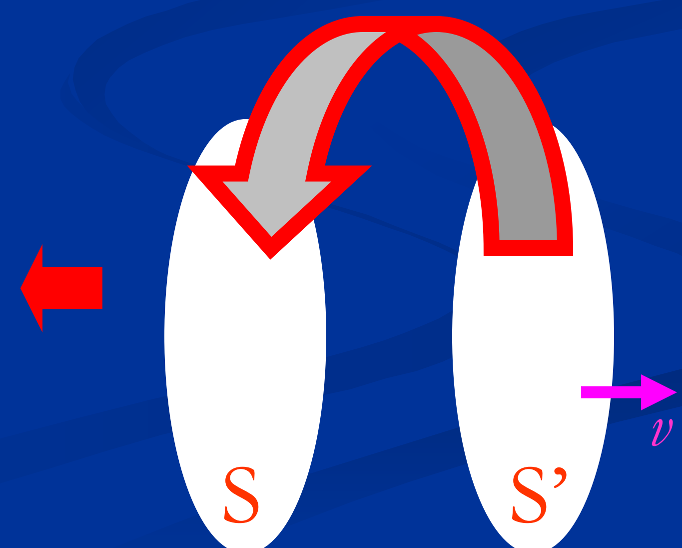


$$x' = x - vt$$
$$y' = y$$
$$z' = z$$
$$t' = t$$

Galilean Transformation Equations

$$x = x' + vt'$$
$$y = y'$$
$$z = z'$$
$$t = t'$$

Inverse Galilean Transformation Equations



Galilean Transformation Equations

$$x' = x - vt$$
$$y' = y$$
$$z' = z$$
$$t' = t$$

Galilean Transformation Equations

$$x = x' + vt'$$
$$y = y'$$
$$z = z'$$
$$t = t'$$

Inverse Galilean Transformation Equations

They enable us to relate a measurement in one inertial reference frame to another. For example, suppose we measure the velocity of a vehicle moving in the x-direction in system S, and we want to know what would be the velocity of the vehicle in S'.

Galilean Velocity Transformation Equations

$$U_x = \frac{dx}{dt}$$

$$U_y = \frac{dy}{dt}$$

$$U_z = \frac{dz}{dt}$$

$$U_x' = \frac{dx'}{dt'}$$

$$U_y' = \frac{dy'}{dt'}$$

$$U_z' = \frac{dz'}{dt'}$$

$$u_x' = u_x - v$$

$$u_y' = u_y$$

$$u_z' = u_z$$

Galilean Velocity Transformation Equations

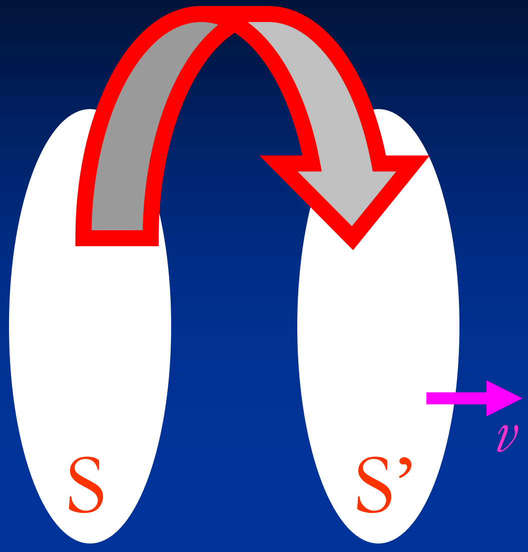
$$u_x = u_x' + v$$

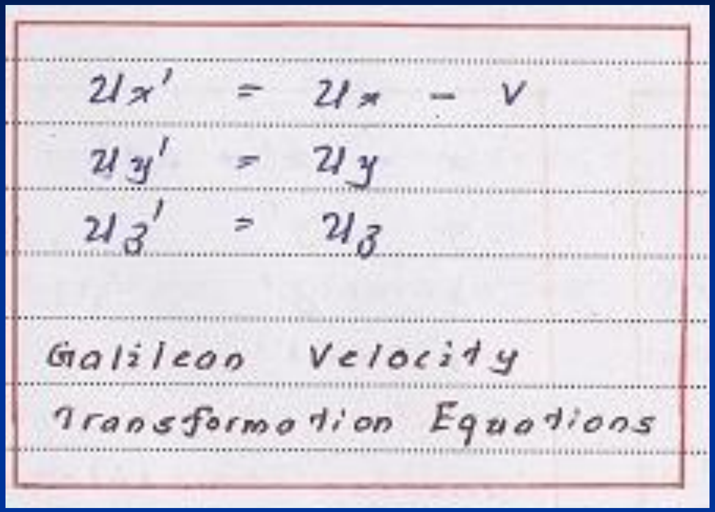
$$u_y = u_y'$$

$$u_z = u_z'$$

Inverse Galilean Velocity Transformation Equations

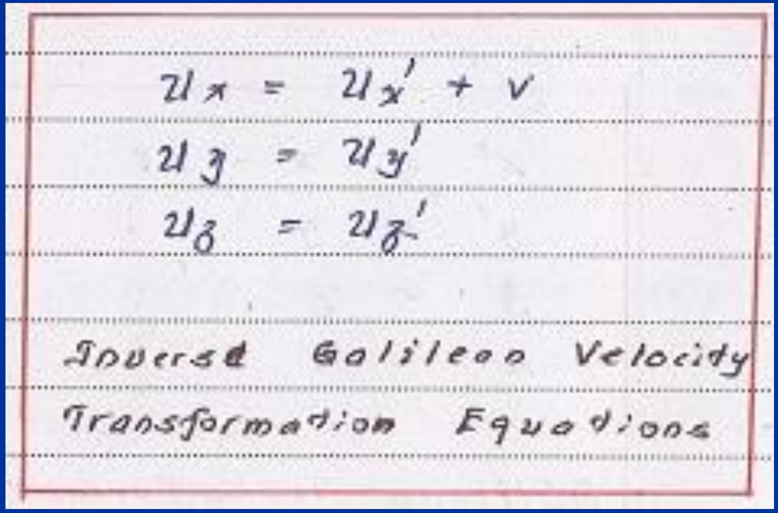
Galilean Velocity Transformation Equations





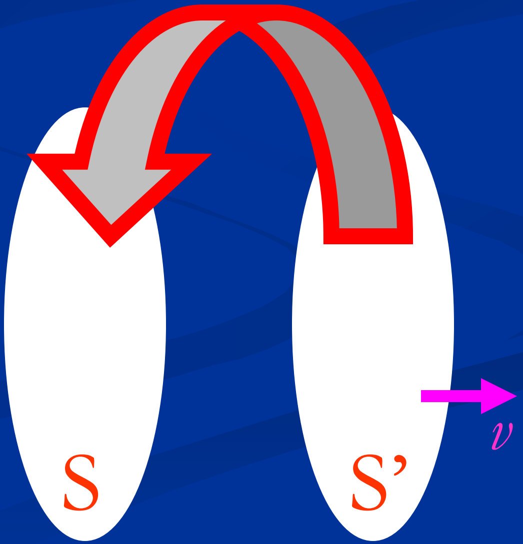
$$u_x' = u_x - v$$
$$u_y' = u_y$$
$$u_z' = u_z$$

Galilean Velocity Transformation Equations



$$u_x = u_x' + v$$
$$u_y = u_y'$$
$$u_z = u_z'$$

Inverse Galilean Velocity Transformation Equations



Galilean Acceleration Transformation Equations

$$a_x = \frac{dU_x}{dt}$$

$$a_y = \frac{dU_y}{dt}$$

$$a_z = \frac{dU_z}{dt}$$

$$a_x' = \frac{dU_x'}{dt'}$$

$$a_y' = \frac{dU_y'}{dt'}$$

$$a_z' = \frac{dU_z'}{dt'}$$

$$a_x' = a_x$$

$$a_y' = a_y$$

$$a_z' = a_z$$

Galileon Accelerator Transformation Equation

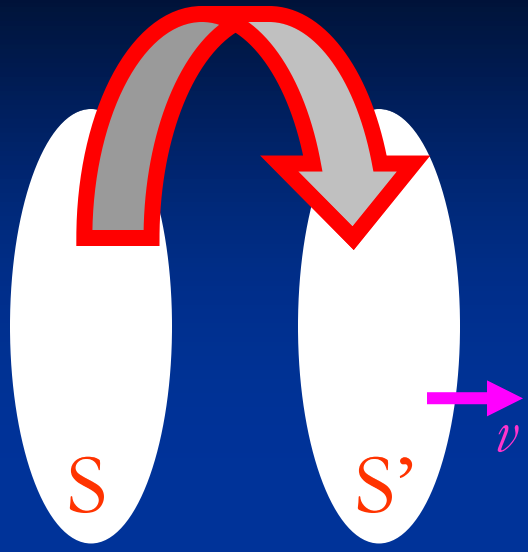
$$a_x = a_x'$$

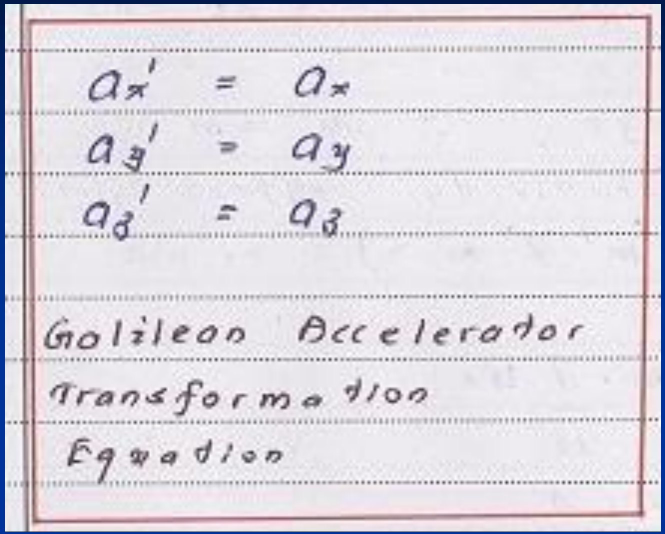
$$a_y = a_y'$$

$$a_z = a_z'$$

Inverse Galileon Accelerator Transformation Equation

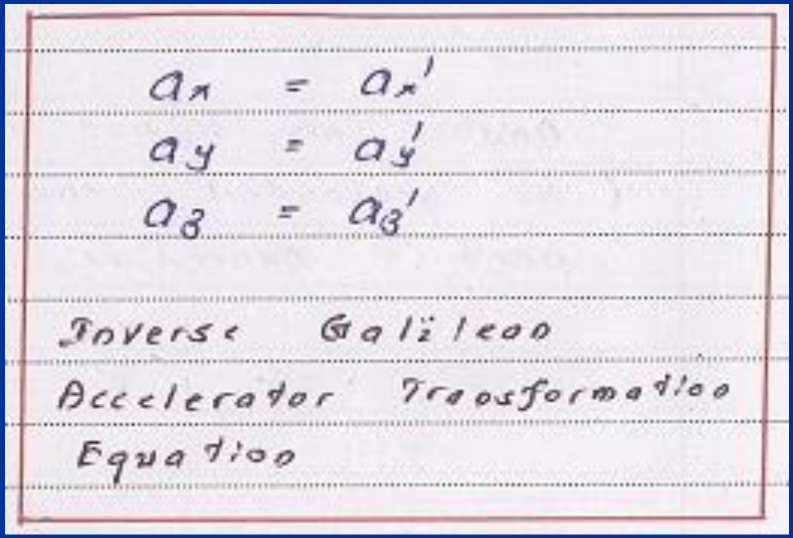
Galilean Acceleration Transformation Equations





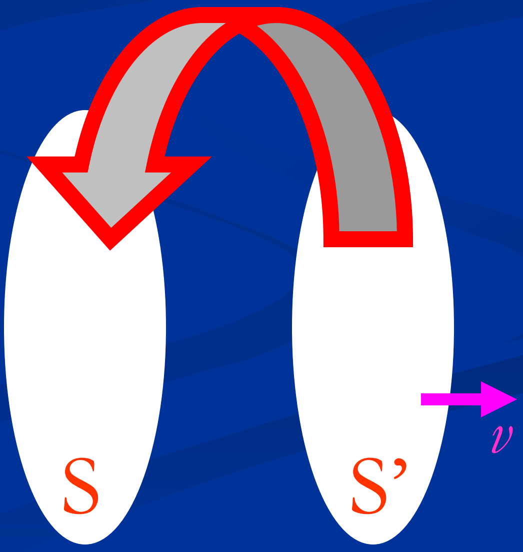
$$a_x' = a_x$$
$$a_y' = a_y$$
$$a_z' = a_z$$

Galilean Accelerator Transformation Equation

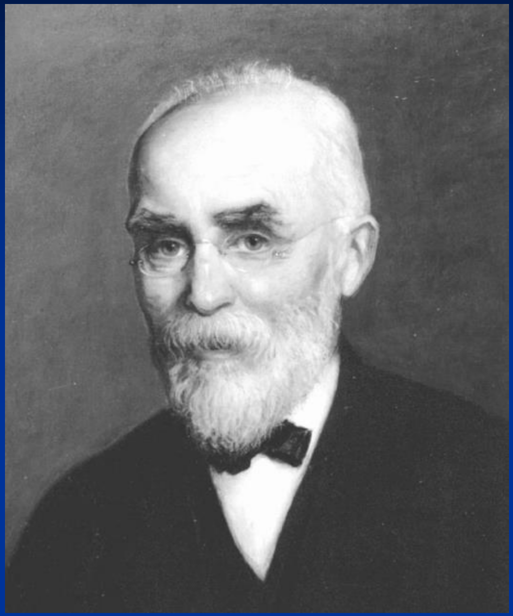


$$a_x = a_x'$$
$$a_y = a_y'$$
$$a_z = a_z'$$

Inverse Galilean Accelerator Transformation Equation

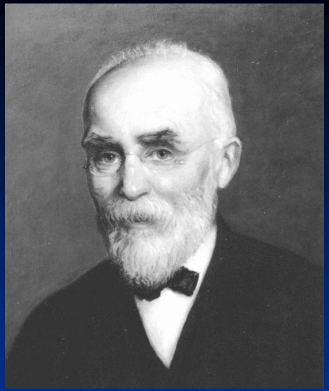


Lorentz Transformation



In physics (1904), the **Lorentz transformation** or **Lorentz-Fitzgerald transformation** describes how, according to the theory of special relativity, different measurements of space and time by two observers can be converted into the measurements observed in either frame of reference.

Lorentz Transformation



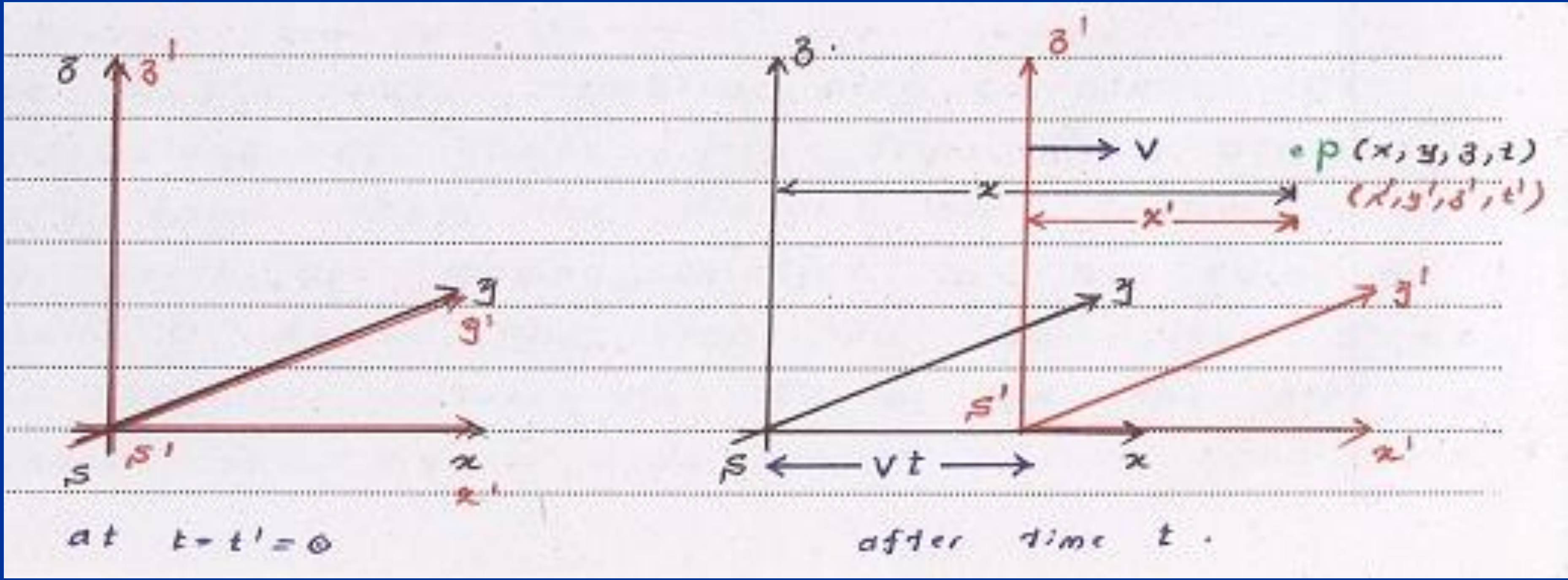
The Lorentz transformation was originally the result of attempts by Lorentz and others to explain how the speed of light was observed to be independent of the reference frame, and to understand the symmetries of the laws of electromagnetism. Albert Einstein later re-derived the transformation from his postulates of special relativity. The Lorentz transformation supersedes the Galilean transformation of Newtonian physics, which assumes an absolute space and time. According to special relativity, the Galilean transformation is a good approximation only at relative speeds much smaller than the speed of light.

Lorentz Transformation Equations

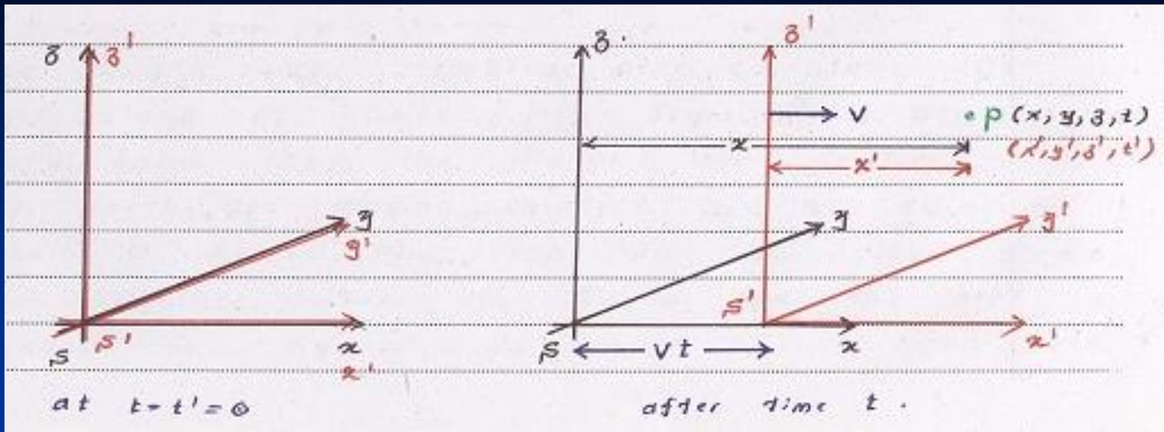
Consider two observers O and O', each using their own Cartesian coordinate system to measure space and time intervals. O uses (t, x, y, z) and O' uses (t', x', y', z'). Assume further that the coordinate systems are oriented so that, in 3 dimensions, the x-axis and the x'-axis are collinear, the y-axis is parallel to the y'-axis, and the z-axis parallel to the z'-axis. The relative velocity between the two observers is v along the common x-axis. Also assume that the origins of both coordinate systems are the same, that is, coincident times and positions.

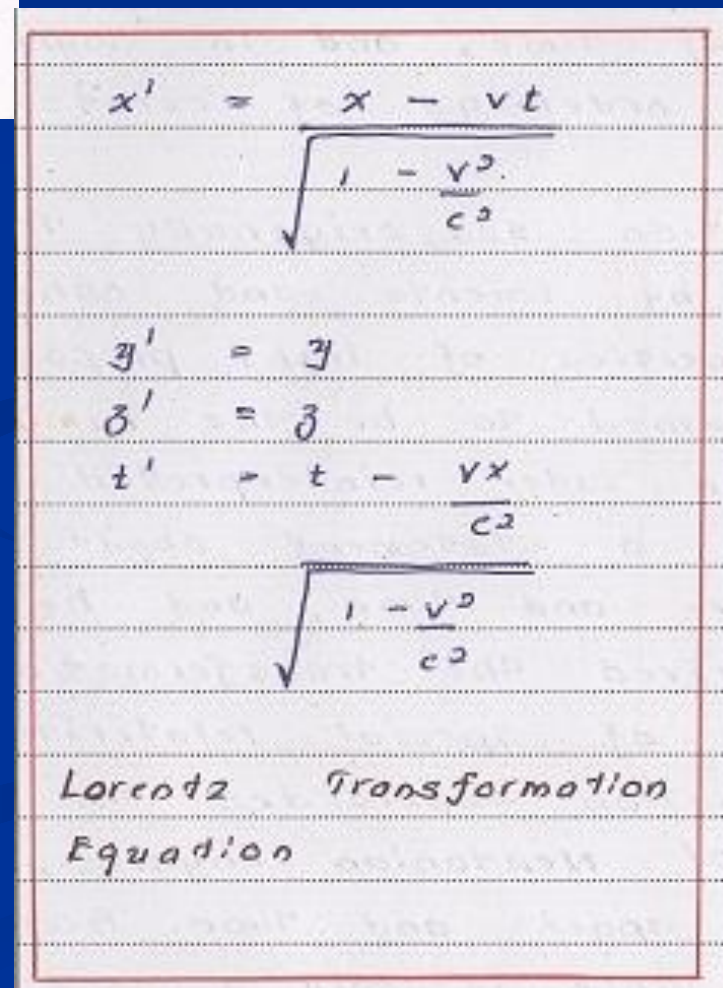
Lorentz Transformation Equations

If all these hold, then the coordinate systems are said to be in **standard configuration**. A between the forward Lorentz Transformation and the inverse Lorentz Transformation can be achieved if coordinate systems are in . The symmetric form highlights that all physical laws should remain unchanged under a Lorentz transformation.



Lorentz Transformation Equations

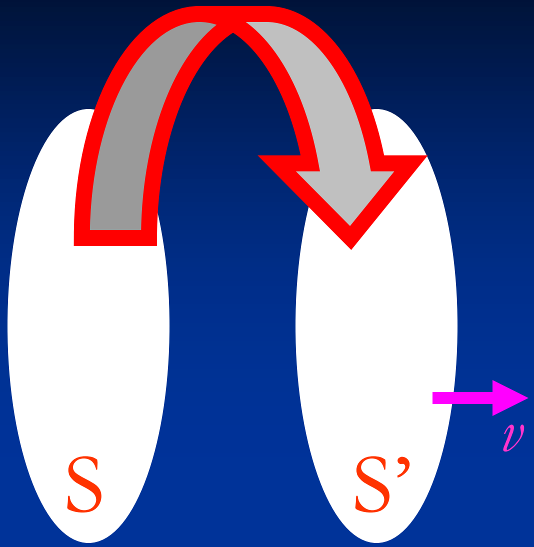




These are the simplest forms. The Lorentz transformation for frames in standard configuration can be shown to be:

where: *v* is the relative velocity between frames in the *x*-direction, 1/(1-v²/c²)^½ , is the Lorentz factor,

Lorentz Transformation Equations



$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$y' = y$$

$$z' = z$$

$$t' = \frac{t - \frac{vx}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Lorentz Transformation Equations

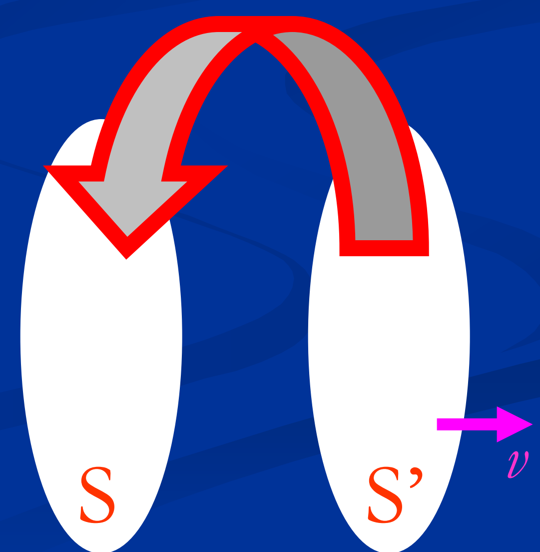
Lorentz Inverse Transformation Equations

$$x = \frac{x' + vt'}{\sqrt{1 - \frac{v^2}{c^2}}}$$

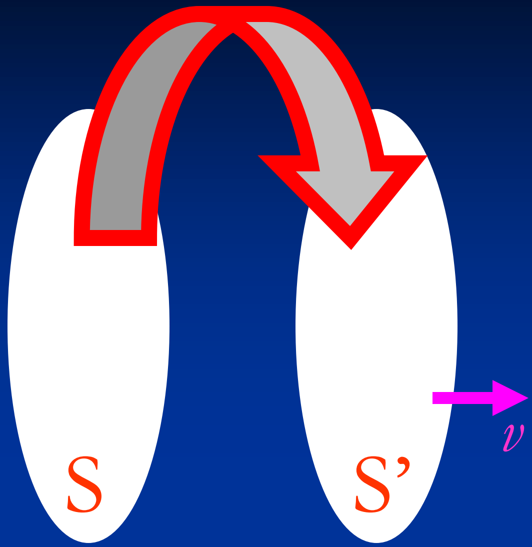
$$y = y'$$

$$z = z'$$

$$t = \frac{t' + \frac{vx'}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$$



Lorentz Velocity Transformation Equations



$$u_x' = \frac{u_x - v}{1 - \frac{v}{c^2} \cdot u_x}$$

$$u_y' = \frac{u_y}{\gamma\left(1 - \frac{v}{c^2} u_x\right)}$$

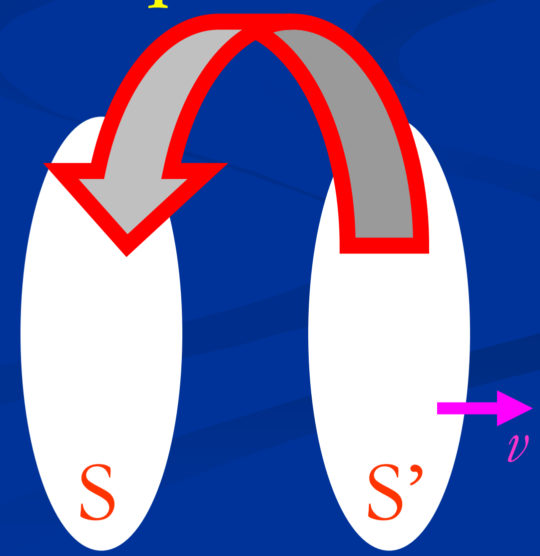
$$u_z' = \frac{u_z}{\gamma\left(1 - \frac{v}{c^2} \cdot u_x\right)}$$

Lorentz Velocity Transformation Equations

$$u_x = \frac{u_x' + v}{1 - \frac{v}{c^2} \cdot u_x'}$$

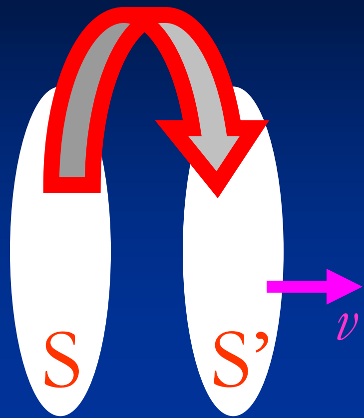
$$u_y = \frac{u_y'}{\gamma\left(1 - \frac{v}{c^2} \cdot u_x'\right)}$$

$$u_z = \frac{u_z'}{\gamma\left(1 - \frac{v}{c^2} u_x'\right)}$$



Lorentz Inverse Velocity Transformation Equations

Lorentz Acceleration Transformation Equations

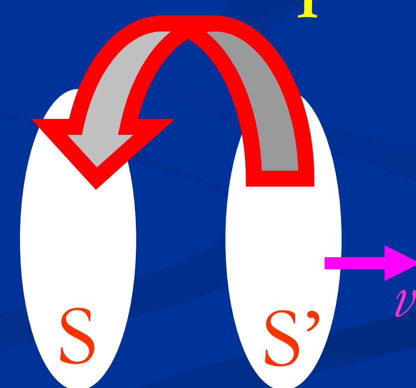


$$a'_x = \frac{a_x}{\gamma^3\left(1 - \frac{v}{c^2}u_x\right)^3}$$

$$a'_y = \frac{a_y\left(1 - \frac{v u_x}{c^2}\right) + \frac{v}{c^2}u_y a_x}{\gamma^2\left(1 - \frac{v}{c^2}u_x\right)^3}$$

$$a'_z = \frac{a_z\left(1 - \frac{v u_x}{c^2}\right) + \frac{v}{c^2}u_z a_x}{\gamma^2\left(1 - \frac{v}{c^2}u_x\right)^3}$$

Lorentz Acceleration Transformation Equations



$$a_x = \frac{a'_x}{\gamma^3\left(1 + \frac{v}{c^2}u'_x\right)^3}$$

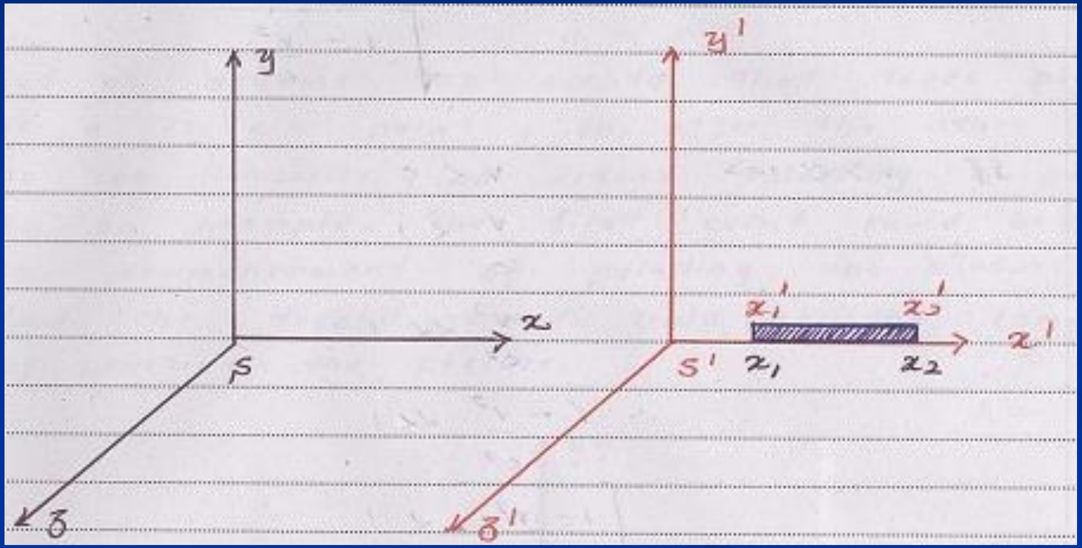
$$a_y = \frac{a'_y\left(1 + \frac{v u'_x}{c^2}\right) - \frac{v}{c^2}u'_y a'_x}{\gamma^2\left(1 + \frac{v}{c^2}u'_x\right)^3}$$

$$a_z = \frac{a'_z\left(1 + \frac{v u'_x}{c^2}\right) - \frac{v}{c^2}u'_z a'_x}{\gamma^2\left(1 + \frac{v}{c^2}u'_x\right)^3}$$

Lorentz Inverse Acceleration Transformation Equations

Length Contraction (using Lorentz Transformation)

Length contraction is the observation that a moving object appears shorter than a stationary object.



Let us assume there is a rod of length L with respect to the stationary frame S’, is moving with constant velocity v.

Length with respect to the S’ frame = Lo = $x_2^1 - x_1^1$

Length with respect to the S frame = L = $x_2 - x_1$

Length Contraction (using Lorentz Transformation)

Using Lorentz transformation equations;

$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$x^1 = \gamma(x - vt)$$

Therefore, $x_1^1 = \gamma(x_1 - vt_1)$ and $x_2^1 = \gamma(x_2 - vt_2)$

$$x_2^1 - x_1^1 = \gamma(x_2 - vt_2) - \gamma(x_1 - vt_1)$$

$$x_2^1 - x_1^1 = \gamma(x_2 - x_1) - \gamma v(t_2 - t_1)$$

Where, $t_1 = t_2$

Therefore, $x_2^1 - x_1^1 = \gamma(x_2 - x_1)$

$$L_o = \gamma L$$

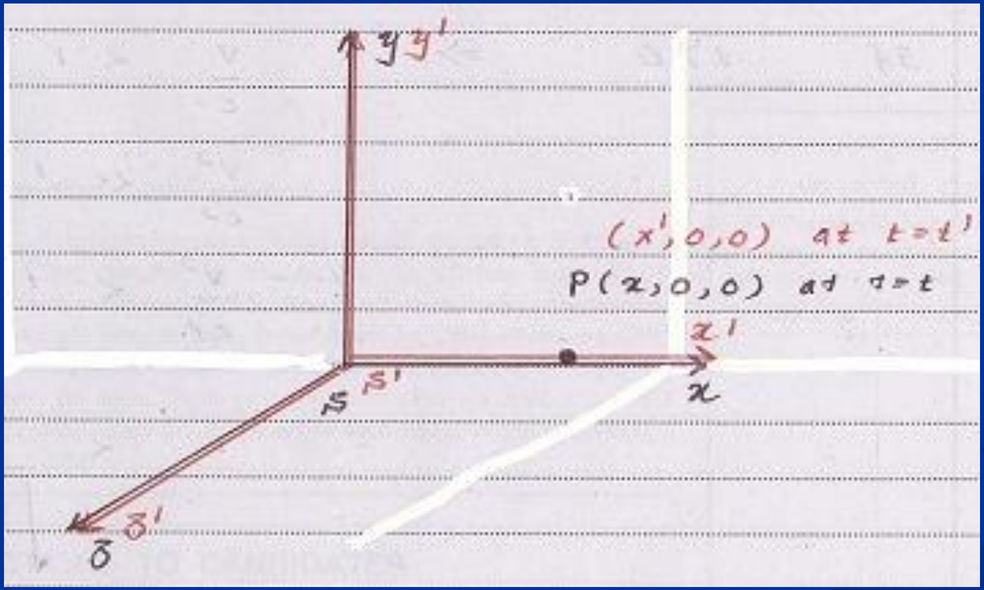
$$L_o > L$$

Length measured by an observer in the Frame S'

Length measured by an observer in the Frame S

Time Dilation (using Lorentz Transformation)

Let us assume two events that takes place at a certain point, one after the other. We can consider, an artist painting a picture as an example. The first event could be the commencement of painting the picture and the second event could be the completion of painting the picture.



Time interval with respect to the S' frame $= t = t_2^1 - t_1^1$

Time interval with respect to the S frame $= t_0 = t_2 - t_1$

Time Dilation (using Lorentz Transformation)

Using Lorentz transformation equations;

$$t' = \frac{t - \frac{vx}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$t^1 = \gamma\left(t - \frac{v}{c^2}x\right)$$

Therefore, $t_1^1 = \gamma\left(t_1 - \frac{v}{c^2}x_1\right); \quad x_1 = x$ and $t_2^1 = \gamma\left(t_2 - \frac{v}{c^2}x_2\right); \quad x_2 = x$

$$t_2^1 - t_1^1 = \gamma\left(t_2 - \frac{v}{c^2}x_2\right) - \gamma\left(t_1 - \frac{v}{c^2}x_1\right)$$

$$t_2^1 - t_1^1 = \gamma(t_2 - t_1) - \gamma\frac{v}{c^2}(x_2 - x_1)$$

Where, $x_1 = x_2$

Therefore, $t_2^1 - t_1^1 = \gamma(t_2 - t_1)$

$$t = \gamma t_0$$

Time interval measured by an observer in the Frame S' ← $t > t_o$ → Time interval measured by an observer in the Frame S

Relative Motion for the two bodies in Relativity

Let us assume two objects are moving in an opposite direction to each other,

Using Lorentz transformation equations;

$$u_x' = \frac{u_x - v}{1 - \frac{v}{c^2} \cdot u_x}$$

For this example; $U_x^1 = V_A$, $U_x = -V_B$ and $v = V_{(B,A)}$

Direction of B is opposite to the A

$$U_x^1 = \frac{U_x - v}{1 - \frac{U_x v}{c^2}} \Rightarrow V_A = \frac{(-V_B) - v}{1 - \frac{(-V_B)v}{c^2}} \Rightarrow -v = \frac{V_A + V_B}{1 + \frac{V_A V_B}{c^2}}$$

$$-v = \frac{V_A + V_B}{1 + \frac{V_A V_B}{c^2}}$$

➡ This v denotes $V(B, A)$. $V(B, A)$ has a negative value. :. The direction of V(B, A) should be the opposite direction. :. V(A, B) is +ve;

$$\rightarrow \quad v = V_{(A,B)} = \frac{V_A + V_B}{1 + \frac{V_A V_B}{c^2}}$$

Or $\leftarrow \quad v = V_{(B,A)}$

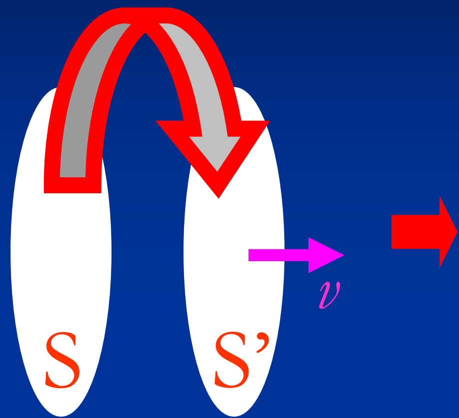
Let us assume two objects are moving in a same direction,

For this example; $U_x^1 = V_A$,

$U_x = V_B$ and $v = V_{(B,A)}$

$$\rightarrow \quad v = V_{(B,A)} = \frac{V_A - V_B}{1 - \frac{V_A V_B}{c^2}}$$

Find the Energy & Momentum in S' frame w. r. t. S frame



$$E^{1} = \gamma(v)\left[E - P_x v\right]$$

$$P_x^{1} = \gamma(v)\left[P_x - \frac{Ev}{c^2}\right],$$

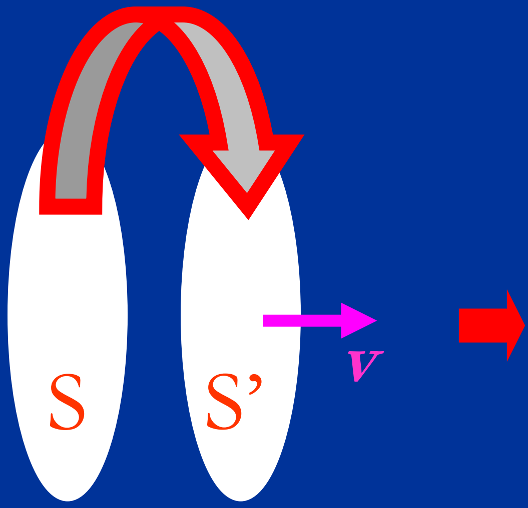
$$P_y^{1} = P_y$$ and $$P_z^{1} = P_z$$

Where, $\gamma(v) = \dfrac{1}{\sqrt{1 - \dfrac{v^2}{c^2}}}$, $E = mc^2 = \dfrac{m_o}{\sqrt{1 - \dfrac{v^2}{c^2}}} c^2 = \gamma(v) m_o c^2$

and $P_x = mU_x = \dfrac{m_o}{\sqrt{1 - \dfrac{v^2}{c^2}}} U_x = \gamma(v) m_o U_x$

Lorentz Invariant

A quantity that remains unchanged by a Lorentz transformation is said to be Lorentz invariant. Such quantities play on especially important role in special theory of relativity. The norm of any four vector is Lorentz Invariant.



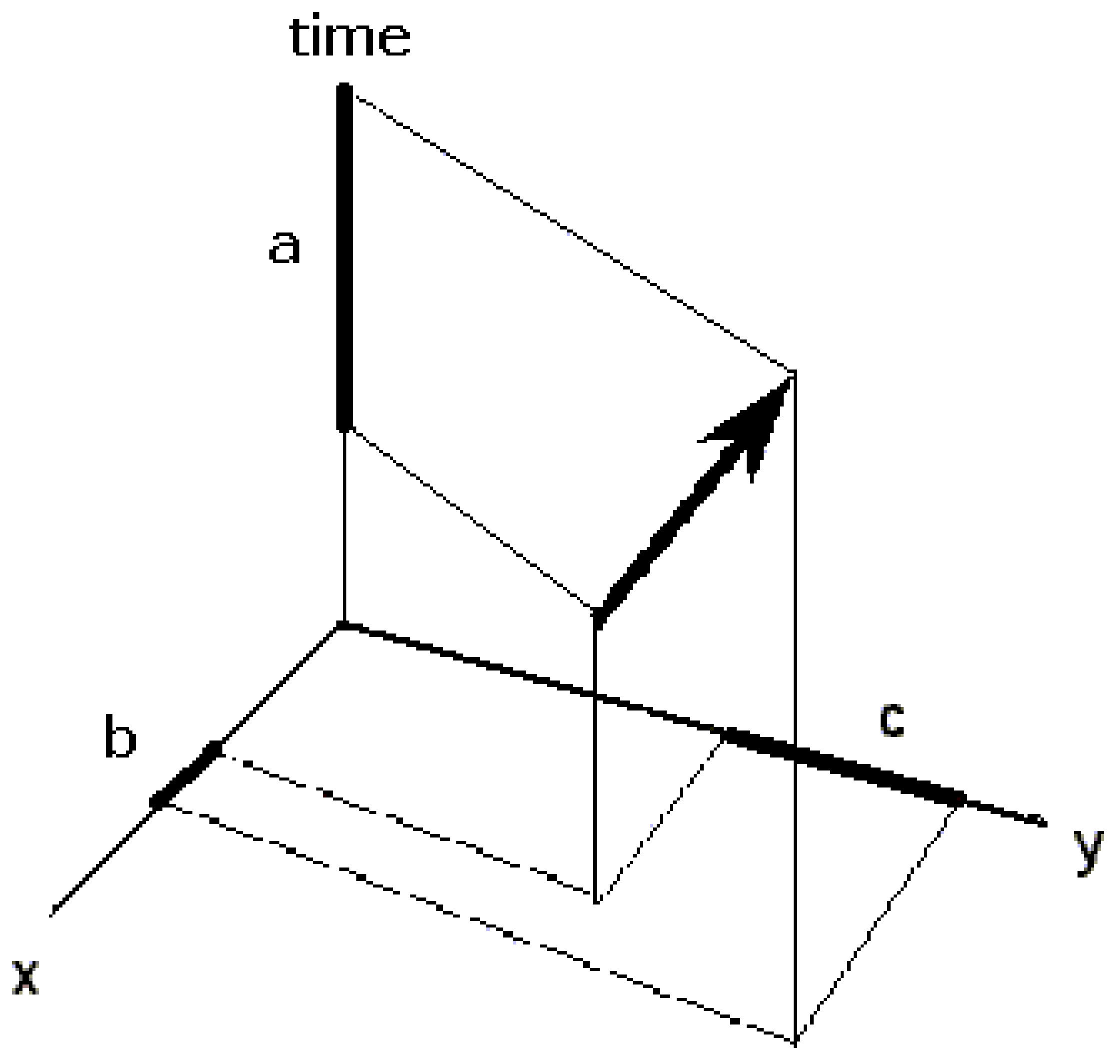
$$E'^2 - p'^2c^2 = E^2 - p^2c^2$$

and

$$c^2t'^2 - r'^2 = c^2t^2 - r^2$$

Four Vectors / Four Cdts Systems

Four vectors



A four vector is a displacement in both space and time.


Thank You !