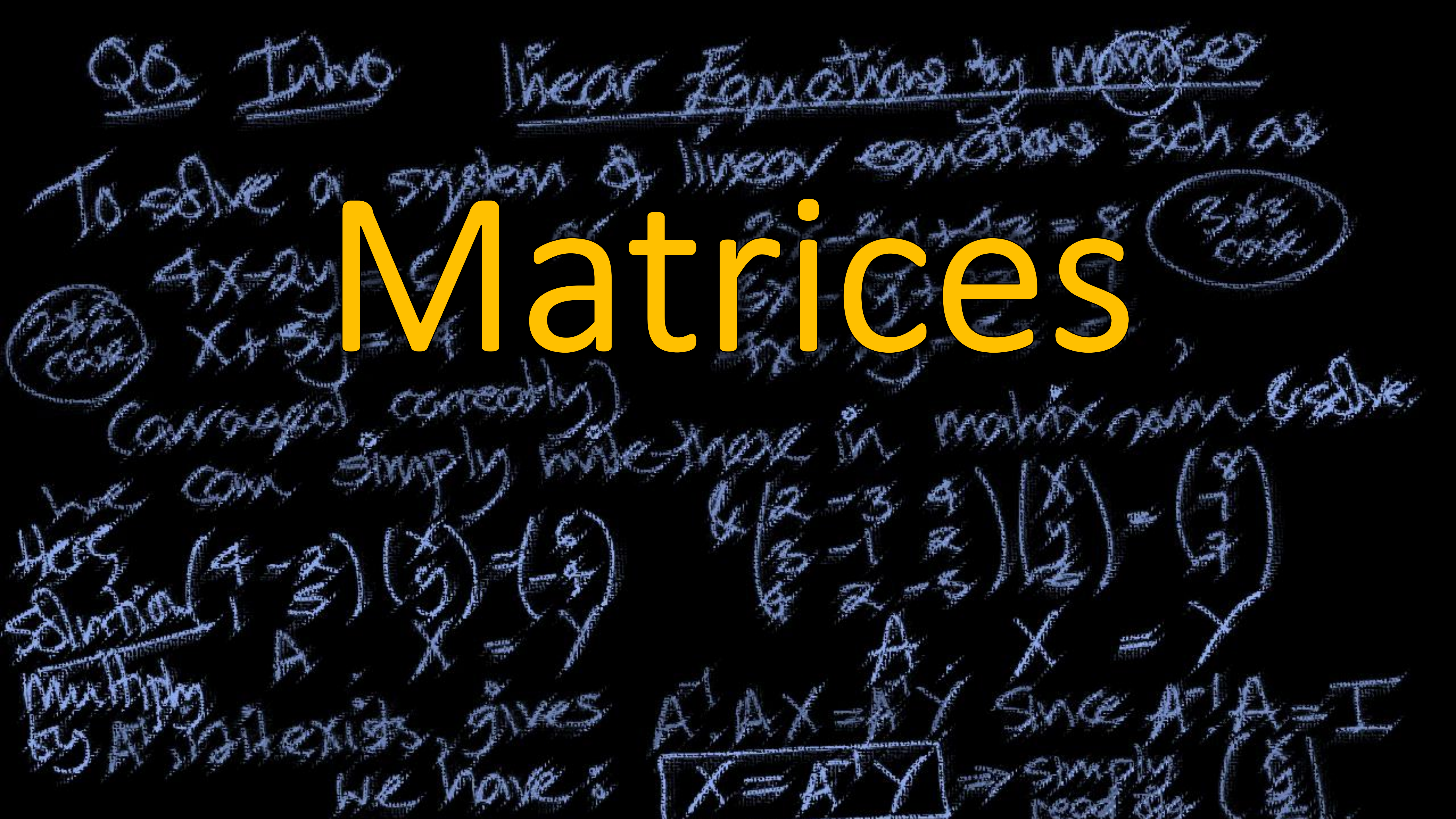
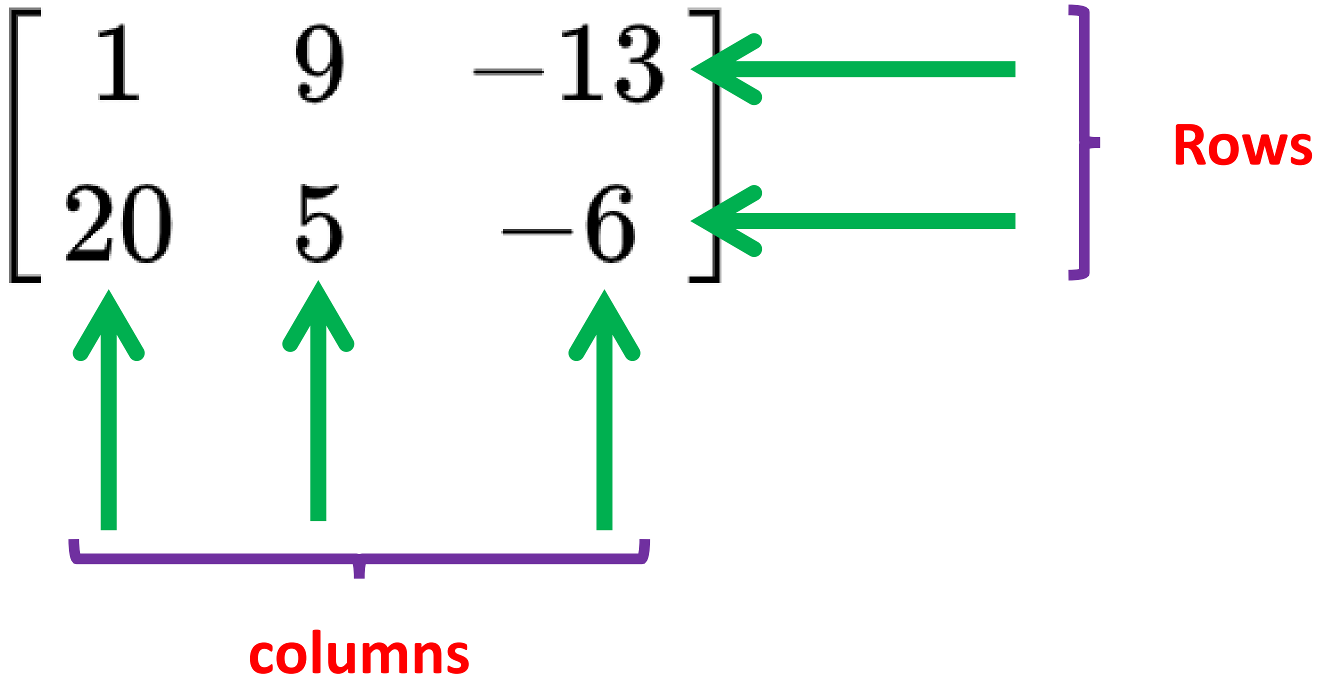
Matrices



Matrix

In mathematics, a matrix (plural matrices) is a rectangular array of **numbers**, **symbols**, or **expressions**, arranged in rows and columns. For example, the dimension of the matrix below is 2 × 3, because there are **two rows** and **three columns**:

$$\begin{bmatrix} 1 & 9 & -13 \\ 20 & 5 & -6 \end{bmatrix}$$



Solving linear simultaneous equations using ***Matrices***

example: Consider the following linear simultaneous equations.

$$x + y + z = 2$$

$$2x - 3y + z = -9$$

$$x - y - 2z = 2$$

1st Step: Convert the equations to matrix form

$$1\,x + 1\,y + 1\,z = 2$$

$$2\,x - 3\,y + 1\,z = -9$$

$$1\,x - 1\,y - 2\,z = 2$$

$$\Rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 2 & -3 & 1 \\ 1 & -1 & -2 \end{pmatrix} . \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -9 \\ 2 \end{pmatrix}$$

1st Step: Convert the equations to matrix form

Coefficient Matrix

Matrix of Unknowns

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & -3 & 1 \\ 1 & -1 & -2 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -9 \\ 2 \end{pmatrix}$$

A X B

(***Dot product*** of the Coefficient matrix and the matrix of unknowns, give the left hand side of the equations)

2nd Step: Take those matrices as follows.

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -3 & 1 \\ 1 & -1 & -2 \end{pmatrix} \quad , \quad \mathrm{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2 \\ -9 \\ 2 \end{pmatrix}$$

According to this, we have

$$A.X = B$$

We have to find X Matrix here…

3rd Step: Rearrange the above equation, By multiplying with A^{-1} from the left hand side.

$$A.X = B$$

Multiply by A^{-1},

$$\underbrace{A^{-1}.A}.X = A^{-1}.B$$

$$\underbrace{I.X} = A^{-1}.B$$

$$X = A^{-1}.B$$

Therefore X can be found by using, $\boxed{X = A^{-1}.B}$

Here, A^{-1} - Inverse Matrix of A and I - Identity Matrix

There are **<u>3 Cases</u>** when solving linear simultaneous equations.
They are:

01. Even Determinant case

no of equations = no of unknowns

02. Over Determinant case / Pre Determinant case

no of equations > no of unknowns

03. Under Determinant case

no of equations < no of unknowns

[In this case we can use Backes & Gilbert Method to solve this]

1. ***Even Determinant case***

In this case ***the Number of Unknowns are equal to the Number of Equations***.
and the Solution can be written as,

$$X = A^{-1}.B$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & -3 & 1 \\ 1 & -1 & -2 \end{pmatrix} . \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -9 \\ 2 \end{pmatrix}$$

Matrices 2020 - Matrix Operations.nb * - Wolfram Mathematica 12.1

File Edit Insert Format Cell Graphics Evaluation Palettes Window Help

```
In[25]:= a = {{1, 1, 1}, {2, -3, 1}, {1, -1, -2}};
         x = {x1, x2, x3};
         b = {2, -9, 2};
         Inverse[a].b // N

Out[28]= {1., 3., -2.}
```

2. ***Over Determinant case / Pre Determinant case***

In this case ***the Number of equations are greater than the Number of Unknowns***.

and the Solution can be written as,

$$X = (A^T.A)^{-1}.(A^T.B)$$

The solution of the Even Determinant case **cannot** be applied here, ***Since the Inverse of Matrix A doesn't exist*** in this case.

[In Pure Mathematics this case will give the same answers as Even Determinant case, but In practical problems, this case is preferred to get the accurate answers for *X* Matrix] example: Consider the following linear simultaneous equations.

$$2x - y + 6z = 1$$
$$5x + 4y + 3z = 0$$
$$9x + 10y + 7z = -1$$
$$11x + 13y + 16z = -10$$
$$5x + 7y + 9z = -19$$

$$\begin{pmatrix} 2 & -1 & 6 \\ 5 & 4 & 3 \\ 9 & 10 & 7 \\ 11 & 13 & 16 \\ 5 & 7 & 9 \end{pmatrix} . \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \\ -10 \\ -19 \end{pmatrix}$$

$$X = (A^T.A)^{-1}.(A^T.B)$$

Matrices 2020 - Matrix Operations.nb * - Wolfram Mathematica 12.1

File Edit Insert Format Cell Graphics Evaluation Palettes Window Help

```
In[29]:= a = {{2, -1, 6}, {5, 4, 3}, {9, 10, 7}, {11, 13, 16}, {5, 7, 9}};
         x = {x1, x2, x3};
         b = {1, 0, -1, -10, -19};
         Inverse[Transpose[a].a].(Transpose[a].b) // N

Out[32]= {3.62383, -2.44044, -1.42036}
```

3. ***Under Determinant case***

In this case ***the Number of equations are less than the Number of Unknowns***.

and the Solution can be written as,

$$X = A^T.(A.A^T)^{-1}.B$$

Using Backes & Gilbert Method

The solution of the Even Determinant case **cannot** be applied here also, ***Since the Inverse of Matrix A doesn't exist*** in this case.

[This case won't give very accurate solutions for Pure Mathematics problems, *But this case is very useful and can be used to get a almost accurate answer for X Matrix, in practical problems*] example: Consider the following linear simultaneous equations.

$$2x - y + 6z = 1$$
$$5x + 4y + 3z = 0$$

$$\Rightarrow \begin{pmatrix} 2 & -1 & 6 \\ 5 & 4 & 3 \end{pmatrix} . \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

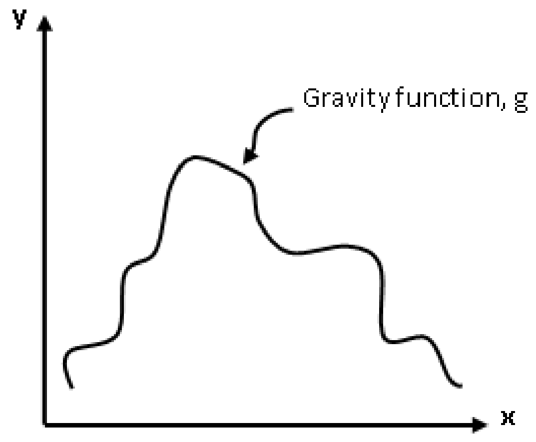
Using Backes & Gilbert Method : $X = A^T.(A.A^T)^{-1}.B$

Matrices 2020 - Matrix Operations.nb * - Wolfram Mathematica 12.1

File Edit Insert Format Cell Graphics Evaluation Palettes Window Help

```
In[37]:= a = {{2, -1, 6}, {5, 4, 3}};
         x = {x1, x2, x3};
         b = {1, 0};
         Transpose[a].Inverse[a.Transpose[a]].(b) // N

Out[40]= {-0.0135685, -0.0990502, 0.154681}
```



Practical Example (1D Case)

If the gravity anomaly is given by the following Equation,

$$g = a\,x^2 + b\,x + c$$

Where ***g* is the gravity** (Measured by gravimeter), ***x* is the distance** *and* ***a*, *b*** *and* ***c*** *are* ***unknown constants***.

[Accuracy of "***g***" is depend on the **gravimeter** used and the accuracy of "***x***" is depend on the **meter ruler** or **GPS System** used]

We have find these ***a***, ***b*** and ***c*** constants.

Gravity is usually measured using the gravimeter (gravity-meter) and therefore the accuracy depends on the gravimeter. Specifically, expensive gravimeters have high accuracy than the cheaper gravimeter. By measuring the gravity at different points, constants *a*, b & c can be accurately determined.

In order to determine the constants *a*, b & c, a student measured the gravity at five points. These readings are given in the table below.

x (m)	Gravity (ms^{-2})
1	1.39
2	2.21
3	3.21
4	4.39
5	5.81

By selecting a suitable method and using the data given in the table, find the values of *a, b* & *c* accurately.

(Values obtained for *a, b* & *c* can be confirmed with the actual values of $a = 0.1$, $b = 0.5$ and $c = 0.8$).

By using the values calculated for *a, b* & *c*, determine the gravity at $x = 2.5$ m and $x = 4.5$ m.

x (m)	Gravity (ms^{-2})
1	1.39
2	2.21
3	3.21
4	4.39
5	5.81

→ $g(x) = a\,x^2 + b\,x + c$

We get,

When

$$x = 1; \quad a + b + c = 1.39 \quad \text{(1)}$$

$$x = 2; \quad 4a + 2b + c = 2.21 \quad \text{(2)}$$

$$x = 3; \quad 9a + 3b + c = 3.21 \quad \text{(3)}$$

$$x = 4; \quad 16a + 4b + c = 4.39 \quad \text{(4)}$$

$$x = 5; \quad 25a + 5b + c = 5.81 \quad \text{(5)}$$

$$\begin{aligned} a + b + c &= 1.39 \\ 4a + 2b + c &= 2.21 \\ 9a + 3b + c &= 3.21 \\ 16a + 4b + c &= 4.39 \\ 25a + 5b + c &= 5.81 \end{aligned}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \\ 16 & 4 & 1 \\ 25 & 5 & 1 \end{pmatrix} . \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1.39 \\ 2.21 \\ 3.21 \\ 4.39 \\ 5.81 \end{pmatrix}$$

$$X = (A^T.A)^{-1}.(A^T.B)$$

Matrices 2020 - Matrix Operations.nb * - Wolfram Mathematica 12.1

File Edit Insert Format Cell Graphics Evaluation Palettes Window Help

```
In[65]:= a = {{1, 1, 1}, {4, 2, 1}, {9, 3, 1}, {16, 4, 1}, {25, 5, 1}};
         x = {aa, bb, cc};
         b = {1.39, 2.21, 3.21, 4.39, 5.81};
         Inverse[Transpose[a].a].(Transpose[a].b) // N
         Print["a = ", %[[1]], "   b = ", %[[2]], "   c = ", %[[3]]]

Out[68]= {0.0985714, 0.510571, 0.786}

         a = 0.0985714   b = 0.510571   c = 0.786
```

Actual values of $a = 0.1$, $b = 0.5$ and $c = 0.8$

Gravity anomaly Equation

$g(x) = a\,x^2 + b\,x + c$ where, $a = 0.098$, $b = 0.510$ & $c = 0.786$

$$g(x) = 0.098\,x^2 + 0.510\,x + 0.786$$

By using the values calculated for *a, b* & *c*, determine the gravity at $x = 2.5$ m and $x = 4.5$ m.

```
Matrices 2020 - Matrix Operations.nb * - Wolfram Mathematica 12.1
File Edit Insert Format Cell Graphics Evaluation Palettes Window Help

In[1]:= Clear[x, g]
        g[x_] = 0.098*x^2 + 0.510*x + 0.786;
        g[2.5];
        Print["The Grav. Ano at x = 2.5 m is   ", g[2.5], " m/(s^2)"]
        g[4.5];
        Print["The Grav. Ano at x = 4.5 m is   ", g[4.5], " m/(s^2)"]

        The Grav. Ano at x = 2.5 m is   2.6735 m/(s^2)

        The Grav. Ano at x = 4.5 m is   5.0655 m/(s^2)
```

Complete Source Code :

Matrices 2020 - Matrix Operations.nb * - Wolfram Mathematica 12.1

File Edit Insert Format Cell Graphics Evaluation Palettes Window Help

```
In[7]:= a = {{1, 1, 1}, {4, 2, 1}, {9, 3, 1}, {16, 4, 1}, {25, 5, 1}};
x = {aa, bb, cc};
b = {1.39, 2.21, 3.21, 4.39, 5.81};
sol = Inverse[Transpose[a].a].(Transpose[a].b) // N;
aa = sol[[1]]; bb = sol[[2]]; cc = sol[[3]];
Print["a = ", aa, "    b = ", bb, "    c = ", cc]
Clear[x, g]
g[x_] = aa*x^2 + bb*x + cc;
Print["The Grav. Ano at x = 2.5 m is    ", g[2.5], " m/(s^2)"]
Print["The Grav. Ano at x = 4.5 m is    ", g[4.5], " m/(s^2)"]

a = 0.0985714    b = 0.510571    c = 0.786

The Grav. Ano at x = 2.5 m is    2.6785 m/(s^2)

The Grav. Ano at x = 4.5 m is    5.07964 m/(s^2)
```

For ***Even Determinant case***

If we only consider Equation No: 1, 2 and 3. we get,

$a = 0.09$, $b = 0.55$ *and* $c = 0.75$

Home Work

For ***Over Determinant case / Pre Determinant case***

If we consider all 5 Equations. we get,

$a = 0.0985714$, $b = 0.510571$ *and* $c = 0.786$

For ***Under Determinant case***

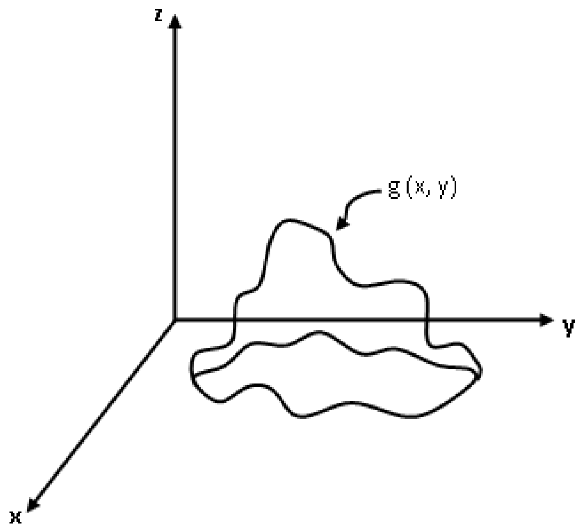
If we only consider Equation No: 1 and 2. we get,

$a = 0.0942857$, $b = 0.537143$ and

$c = 0.758571$

Home Work

Practical Example (2D Case)



If the gravity anomaly is given by the following Equation,

$$g\,(x, y) = a\,x^2 + b\,y^2 + c\,y + d,$$

Where ***g*** ***is the gravity*** (Measured by gravimeter), ***x*** ***and*** ***y*** ***is the distance*** *and* ***a, b, c*** *and* ***d*** *are* ***unknown constants***.

[Accuracy of "***g***" is depend on the **gravimeter** used and the accuracy of "***x***" *and* "***y***" is depend on the **meter ruler** or **GPS System** used]

We have find these ***a***, ***b, c*** and ***d*** constants.

By measuring the gravity at different points, constants *a*, b, c & d can be accurately determined.

In order to determine the constants *a*, b, c & d, a student measured the gravity at eight points. These readings are given in the table below.

x (m)	y (m)	Gravity (ms^{-2})
0	0	0.95
0	1	1.19
1	0	1.09
1	1	1.21
2	0	1.39
0	2	1.81
2	1	1.40
1	2	1.72

By selecting a suitable method and using the data given in the table, find the values of a, b, c & d accurately. (Values obtained for a, b, c & d can be confirmed with the actual values of a = 0.1, b = 0.2, c = -0.1 and d =1.0).

By using the values calculated for a, b, c & d determine the gravity at x = 1.5 m, y = 1.25 m and x = 0.25 m, y = 1.75 m.

x (m)	y (m)	Gravity (ms^{-2})
0	0	0.95
0	1	1.19
1	0	1.09
1	1	1.21
2	0	1.39
0	2	1.81
2	1	1.40
1	2	1.72

$$g\,(x,\,y) = a\,x^2 + b\,y^2 + c\,y + d$$

Matrices 2020 - Matrix Operations.nb * - Wolfram Mathematica 12.1

File Edit Insert Format Cell Graphics Evaluation Palettes Window Help

```
In[175]:= Clear[a, b, c, d, x, y, g]
          g[x_, y_] = a*x^2 + b*y^2 + c*y + d;
          g[0, 0]
          g[0, 1]
          g[1, 0]
          g[1, 1]
          g[2, 0]
          g[0, 2]
          g[2, 1]
          g[1, 2]

Out[177]= d
Out[178]= b + c + d
Out[179]= a + d
Out[180]= a + b + c + d
Out[181]= 4 a + d
Out[182]= 4 b + 2 c + d
Out[183]= 4 a + b + c + d
Out[184]= a + 4 b + 2 c + d
```

Out[177]= d

Out[178]= b + c + d

Out[179]= a + d

Out[180]= a + b + c + d

Out[181]= 4 a + d

Out[182]= 4 b + 2 c + d

Out[183]= 4 a + b + c + d

Out[184]= a + 4 b + 2 c + d

$x = 0$ & $y = 0$; $0\,a + 0\,b + 0\,c + 1\,d = 0.95$ ← 1

$x = 0$ & $y = 1$; $0\,a + 1\,b + 1\,c + 1\,d = 1.19$ ← 2

$x = 1$ & $y = 0$; $1\,a + 0\,b + 0\,c + 1\,d = 1.09$ ← 3

$x = 1$ & $y = 1$; $1\,a + 1\,b + 1\,c + 1\,d = 1.21$ ← 4

$x = 2$ & $y = 0$; $4\,a + 0\,b + 0\,c + 1\,d = 1.39$ ← 5

$x = 0$ & $y = 2$; $0\,a + 4\,b + 2\,c + 1\,d = 1.81$ ← 6

$x = 2$ & $y = 1$; $4\,a + 1\,b + 1\,c + 1\,d = 1.40$ ← 7

$x = 1$ & $y = 2$; $1\,a + 4\,b + 2\,c + 1\,d = 1.72$ ← 8

$0\,a + 0\,b + 0\,c + 1\,d = 0.95$

$0\,a + 1\,b + 1\,c + 1\,d = 1.19$

$1\,a + 0\,b + 0\,c + 1\,d = 1.09$

$1\,a + 1\,b + 1\,c + 1\,d = 1.21$

$4\,a + 0\,b + 0\,c + 1\,d = 1.39$

$0\,a + 4\,b + 2\,c + 1\,d = 1.81$

$4\,a + 1\,b + 1\,c + 1\,d = 1.40$

$1\,a + 4\,b + 2\,c + 1\,d = 1.72$

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 4 & 0 & 0 & 1 \\ 0 & 4 & 2 & 1 \\ 4 & 1 & 1 & 1 \\ 1 & 4 & 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0.95 \\ 1.19 \\ 1.09 \\ 1.21 \\ 1.39 \\ 1.81 \\ 1.40 \\ 1.72 \end{pmatrix}$$

$$X = (A^T.A)^{-1}.(A^T.B)$$

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 4 & 0 & 0 & 1 \\ 0 & 4 & 2 & 1 \\ 4 & 1 & 1 & 1 \\ 1 & 4 & 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0.95 \\ 1.19 \\ 1.09 \\ 1.21 \\ 1.39 \\ 1.81 \\ 1.40 \\ 1.72 \end{pmatrix}$$

Matrices 2020 - Matrix Operations.nb * - Wolfram Mathematica 12.1

File Edit Insert Format Cell Graphics Evaluation Palettes Window Help

```
In[185]:= a = {{0, 0, 0, 1}, {0, 1, 1, 1}, {1, 0, 0, 1}, {1, 1, 1, 1}, {4, 0, 0, 1},
             {0, 4, 2, 1}, {4, 1, 1, 1}, {1, 4, 2, 1}};
          x = {aa, bb, cc, dd};
          b = {0.95, 1.19, 1.09, 1.21, 1.39, 1.81, 1.40, 1.72};
          sol = Inverse[Transpose[a].a].(Transpose[a].b) // N;
          aa = sol[[1]]; bb = sol[[2]]; cc = sol[[3]];
          dd = sol[[3]];
          Print["a = ", aa, "    b = ", bb, "    c = ", cc, "    d = ", dd]

          a = 0.0765421    b = 0.23215    c = -0.108816    d = -0.108816
```

Actual values of *a = 0.1*, *b = 0.2*, *c = -0.1* and *d =1.0*

By using the values calculated for *a, b, c & d* determine the gravity at $x = 1.5$ m, $y = 1.25$ m and $x = 0.25$ m, $y = 1.75$ m.

Matrices 2020 - Matrix Operations.nb * - Wolfram Mathematica 12.1

File Edit Insert Format Cell Graphics Evaluation Palettes Window Help

```
In[198]:= Clear[x, y, g]
g[x_, y_] = 0.076 * x^2 + 0.232 * y^2 - 0.108 * y - 0.108;
g[1.5, 1.25];
Print["The Grav. Ano at (1.5,1.25)m is   ", g[1.5, 1.25], " m/(s^2)"]
g[0.25, 1.75];
Print["The Grav. Ano at (0.25,1.75)m is   ", g[0.25, 1.75],
 " m/(s^2)"]

The Grav. Ano at (1.5,1.25)m is   0.2905 m/(s^2)

The Grav. Ano at (0.25,1.75)m is   0.41825 m/(s^2)
```

Complete Source Code :

Matrices 2020 - Matrix Operations.nb * - Wolfram Mathematica 12.1

File Edit Insert Format Cell Graphics Evaluation Palettes Window Help

```
In[204]:= a = {{0, 0, 0, 1}, {0, 1, 1, 1}, {1, 0, 0, 1}, {1, 1, 1, 1}, {4, 0, 0, 1},
             {0, 4, 2, 1}, {4, 1, 1, 1}, {1, 4, 2, 1}};
           x = {aa, bb, cc, dd};
           b = {0.95, 1.19, 1.09, 1.21, 1.39, 1.81, 1.40, 1.72};
           sol = Inverse[Transpose[a].a].(Transpose[a].b) // N;
           aa = sol[[1]]; bb = sol[[2]]; cc = sol[[3]]; dd = sol[[3]];
           Print["a = ", aa, "    b = ", bb, "    c = ", cc, "    d = ", dd]
           Clear[x, y, g]
           g[x_, y_] = 0.076 * x^2 + 0.232 * y^2 - 0.108 * y - 0.108;
           Print["The Grav. Ano at (1.5,1.25)m is   ", g[1.5, 1.25], " m/(s^2)"]
           Print["The Grav. Ano at (0.25,1.75)m is   ", g[0.25, 1.75], " m/(s^2)"]

a = 0.0765421    b = 0.23215    c = -0.108816    d = -0.108816

The Grav. Ano at (1.5,1.25)m is   0.2905 m/(s^2)

The Grav. Ano at (0.25,1.75)m is   0.41825 m/(s^2)
```

Thank you !