Knowing $I(\theta)$ we can find the *flux density* F, i.e., the number of photons, or better, the total energy which crosses *at right angles* a unit area per unit time. From the definition of F it follows that,

$$F = \iint I(\theta) \cos\theta \, d\Omega = 2\pi \int I(\theta) \cos\theta \sin\theta \, d\theta \qquad \text{(A-2)}$$

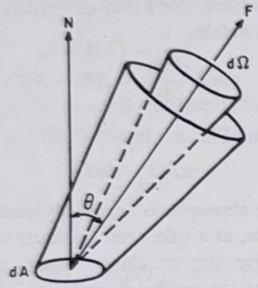

FIGURE A.1-I The geometry which defines the Specific Intensity $I_\nu(\theta)$

The momentum crossing per unit time a unit area at an angle θ to the normal is equal to the flux in this direction divided by the speed of light c. Hence it is equal to $I(\theta) \cos\theta \, d\Omega/c$, and its component normal to the surface will be $I(\theta) \cos^2\theta \, d\Omega/c$. But the time change of momentum per unit area is pressure, and therefore, integrating over all angles we obtain the *radiation pressure P*,

$$P = \frac{1}{c} \iint I(\theta) \cos^2\theta \, d\Omega = \frac{2\pi}{c} \int I(\theta) \cos^2\theta \sin\theta \, d\theta \qquad \text{(A-3)}$$

In (A-1), (A-2), and (A-3) we have averaged $I(\theta)$, $I(\theta)\cos\theta$, and $I(\theta)\cos^2\theta$ over a whole sphere. When $I(\theta)$ is independent of the angle θ, i.e., for isotropic radiation where $I(\theta) = I$, we simply find that $J = I$, $F = 0$, and $P = \dfrac{4\pi I}{3c}$. This, however, can not be exactly the case in a planetary or stellar atmosphere, because $F = 0$, i.e., zero net flux means that no radiation is advancing toward the top of the atmosphere and therefore no radiation will be emitted from the top of the atmosphere.

The next simplest approximation is to assume that $I(\theta)$ has only two constant values, one for all the upward moving radiation, so that for $0 < \theta < 90°$,

$$I(\theta) = I_u \qquad \text{(A-4)}$$

and one for all the downward moving radiation, so that for $90° < \theta < 180°$,

$I(\theta) = I_d$

This approximation (Figure A.1-III) has been very useful in dealing with problems of radiation in stellar and planetary atmospheres. Using (A-4) and (A-5), in (A-1), (A-2), and (A-3) we get,

$$J = \frac{1}{2}(I_u + I_d) \qquad \text{(A-6)}$$

$$F = \pi(I_u - I_d) = F_u - F_d \qquad \text{(A-7)}$$

$$P = \frac{2\pi}{3c}(I_u + I_d) = \frac{4\pi}{3c} J \qquad \text{(A-8)}$$

The Equation of Radiative Transfer

Let us now consider a beam of radiation transversing a layer of thickness dh at an angle θ to the vertical (Figure A.1-II). The intensity $I_\nu(\theta)$ of the radiation will be reduced by absorption in the layer and will be enhanced by emission from the medium. The amount absorbed per unit volume is proportional to $I_\nu(\theta)$ and to the density ϱ of the medium. The constant of proportionality $\varkappa_\nu$ is called the *mass absorption coefficient*. The amount emitted per unit volume, on the other hand, is equal to the *mass emission coefficient* j_ν times the density ϱ of the medium. Since the distance transversed by the beam inside the layer is $dh/\cos\theta$, the intensity of the radiation after crossing the layer at an angle θ to the normal will change by an amount $dI_\nu(\theta)$ where,

$$dI_\nu(\theta) = -I_\nu(\theta)\, \varrho \varkappa_\nu \frac{dh}{\cos\theta} + j_\nu \varrho \frac{dh}{\cos\theta} \qquad \text{(A-9)}$$

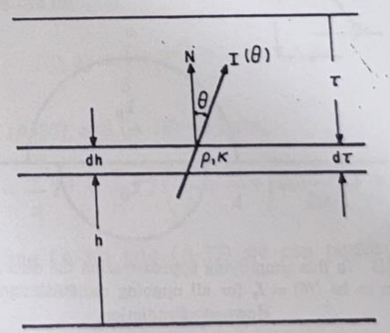

FIGURE A.1-II A beam of radiation transversing an atmospheric layer of thickness dh at an angle θ to the vertical

It is now convenient to introduce the *opacity* τ_ν of the medium, which sometimes is also called the *optical thickness* or the *optical depth* of the medium. The opacity τ_ν is defined by the relation,

$$d\tau_\nu = -\varrho\varkappa_\nu\, dh \quad \text{(A-10)}$$

where the minus sign is used because the optical depth is measured downwards from the top of the atmosphere where $\tau_\nu = 0$ (Figure A.1-II). Using (A-10) in (A-9) we get,

$$\cos\theta \frac{dI_\nu(\theta)}{d\tau_\nu} = I_\nu(\theta) - \frac{j_\nu}{\varkappa_\nu} \quad \text{(A-11)}$$

The atmospheres of the stars and the planets are usually in *local thermodynamic equilibrium* (LTE) which means that the average intensity of radiation at each point of the atmosphere is equal to the emission of a black-body at the local temperature T. Under LTE, $\varkappa_\nu$ and j_ν are related through Kirchhoff's law,

$$\frac{j_\nu}{\varkappa_\nu} = B_\nu(T) \quad \text{(A-12)}$$

where $B_\nu(T)$ is the intensity of the black-body radiation,

$$B_\nu(T)\, d\nu = \frac{2h\nu^3}{c^2} \frac{1}{e^{(h\nu/kT)} - 1}\, d\nu \quad \text{(A-13)}$$

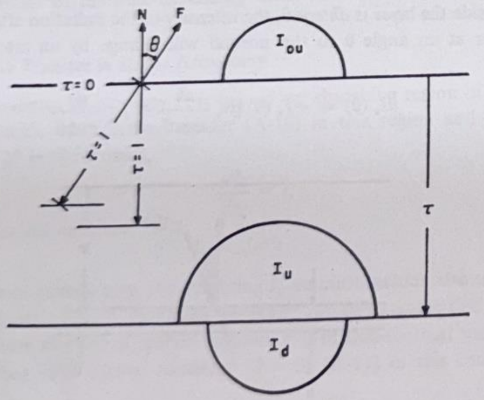

FIGURE A.1-III In this simplifying approximation the directional dependance of $I(\theta)$ is taken to be $I(\theta) = I_u$ for all upgoing radiation and $I(\theta) = I_d$ for all downgoing radiation

Thus under LTE the equation of radiative transfer becomes,

$$\cos\theta \frac{dI_\nu(\theta)}{d\tau_\nu} = I_\nu(\theta) - B_\nu(T) \quad \text{(A-14)}$$

Note that since black-body radiation is isotropic, $B_\nu(T)$ is not a function of θ as is $I_\nu(\theta)$.

When the absorption coefficient is independent of the wavelength, we say that we have *gray matter*. For a *gray atmosphere* we can integrate (A-14) over all frequencies to obtain,

$$\cos\theta \frac{dI(\theta)}{d\tau} = I(\theta) - B(T) \quad \text{(A-15)}$$

where, as we know from Planck's law,

$$\pi B(T) = \sigma T^4 \quad \text{(A-16)}$$

Stellar and planetary atmospheres are usually treated as plane, horizontally stratified layers because, as a rule, their thickness is much smaller than the radius of the respective star or planet. Energy moves from the lowest layers to the top of the atmosphere from which it is radiated into the surrounding space. The energy in most cases is transferred from one atmospheric layer to the next through radiation, rather than through conduction or convection. This is called *radiative transfer*. When an atmosphere is in *radiative equilibrium* the flux remains the same at all depths because otherwise we would have accumulation of energy at certain layers. This means that in a horizontally stratified atmosphere,

$$\frac{dF}{d\tau} = 0 \quad \text{(A-17)}$$

It should be noted here that only the total energy flux remains constant, and radiative equilibrium does not imply that $dF_\nu/d\tau_\nu$ is also equal to zero. The constant flux F, which passes continually through the different layers and is ultimately emitted from the top of the atmosphere, determines the *effective temperature* T_e of the star or the planet. T_e is the temperature at which a black-body would emit the same flux. Hence T_e is defined through the relation,

$$F = \sigma T_e^4 \quad \text{(A-18)}$$

The Eddington Approximation

We will now try to solve the equation of radiative transfer (A-15) using the approximation of (A-4) and (A-5) and assuming, as Eddington did, that at large optical depths $I_u \simeq I_d$. By integrating (A-15) over all angles and using (A-1) and (A-2) we get,

$$\frac{dF}{d\tau} = 4\pi J - 4\pi B(T) \tag{A-19}$$

Furthermore, assuming that the atmosphere is in radiative equilibrium, where as seen from (A-17) $dF/d\tau = 0$, we find from (A-19) that,

$$J = B(T) = \frac{\sigma}{\pi} T^4 \tag{A-20}$$

Next, we can multiply (A-15) by $\cos\theta$ and then integrate it over all angles. The result, using (A-2) and (A-3), is,

$$c\frac{dP}{d\tau} = F \tag{A-21}$$

because,

$$\iint B(T)\cos\theta \, d\Omega = 2\pi B(T)\int_0^\pi \cos\theta \sin\theta \, d\theta = 0 \tag{A-22}$$

Since, as we have seen, F is constant with respect to τ we can integrate (A-21) to obtain,

$$cP = F\tau + C_0 \tag{A-23}$$

where C_0 is the integration constant. Replacing now P with J from (A-8) we get,

$$\frac{4\pi}{3} J = F\tau + C_0 \tag{A-24}$$

At the very top of the atmosphere (Figure A.1-III) there is no downward moving radiation because above the top of the atmosphere there is only the free space. As a result, from (A-6) and (A-7) we have that at $\tau = 0$, $J_0 = \frac{1}{2} I_{0u}$ and $F_0 = \pi I_{0u}$. Hence setting $\tau = 0$ in (A-24) we get,

$$C_0 = \frac{4\pi}{3} J_0 = \frac{2\pi}{3} I_{0u} = \frac{2}{3} F_0 \tag{A-25}$$

But F is the same at all layers hence F_0 is the same as F and thus (A-24) becomes,

$$\frac{4\pi}{3} J = F\tau + \frac{2}{3} F \tag{A-26}$$

or,

$$J = \frac{F}{\pi}\left(\frac{1}{2} + \frac{3}{4}\tau\right) \tag{A-27}$$

Since at $\tau = 0$ we have $I_d = 0$, and at large optical depths we expect to have nearly isotropic conditions, i.e., at $\tau \gg 1$ we have $I_u \simeq I_d$, it follows from (A-27) and (A-6) that,

$$I_d = \frac{F}{\pi}\left(\frac{3}{4}\tau\right) \tag{A-28}$$

and

$$I_u = \frac{F}{\pi}\left(1 + \frac{3}{4}\tau\right) \tag{A-29}$$

Using now (A-18) and (A-20), we can express (A-27) in terms of T and T_e. The final result is,

$$T = T_e\left(\frac{1}{2} + \frac{3}{4}\tau\right)^{1/4} \tag{A-30}$$

This is the famous *Eddington approximation* which describes the change of the temperature T_e with the optical depth τ. Note that the effective temperature T_e occurs at $\tau = 2/3$ and not at $\tau = 0$. At the top of the atmosphere, i.e. at $\tau = 0$, the temperature T_0 is,

$$T_0 = T_e\left(\frac{1}{2}\right)^{1/4} = 0.86 T_e \tag{A-31}$$

Now that we have obtained the relation between T and τ, we can use it to solve the equation of radiative transfer (A-15). The integral solution of this differential equation is,

$$I(\theta, \tau) = e^{\tau \sec\theta} \int_\tau^\infty B(T)\, e^{-\tau' \sec\theta} \sec\theta \, d\tau' \tag{A-32}$$

which for $\tau = 0$, i.e., for the radiation coming out from the top of the atmosphere, yields the integral,

$$I(\theta, 0) = \int_0^\infty B(T)\, e^{-\tau' \sec\theta} \sec\theta \, d\tau' \tag{A-33}$$

But from (A-20), (A-30), and (A-18) we have,

$$B(T) = \frac{\sigma}{\pi} T^4 = \frac{\sigma}{\pi} T_e^4 \left(\frac{1}{2} + \frac{3}{4}\tau\right) = \frac{F}{2\pi}\left(1 + \frac{3}{2}\tau\right) \tag{A-34}$$

and by introducing (A-34) into (A-33) we can perform the integration to obtain the expression,

$$I(\theta, 0) = \frac{F}{2\pi}\left(1 + \frac{3}{2}\cos\theta\right) \tag{A-35}$$

This final result was derived using the Eddington approximation and assuming local thermodynamic equilibrium, radiative equilibrium, and a horizontally stratified gray atmosphere. The fact that we have used a step-function dependance of I on θ to derive (A-35) should not be considered as a contradiction or as an inconsistency. It is only like using simpler tools to construct others that are more complex and more precise.

From a pragmatic point of view the directional dependence of (A-35) reflects the fact that the radiation emitted in the different directions originates essentially at different depths in the atmosphere. The bulk of the radiation emitted from the top of the atmosphere originates at an optical depth $\tau = 1$ below the top of the atmosphere. As seen from Figure A.1-III, the point which yields an opacity equal to unity occurs closer to the top of the atmosphere for rays propagating at larger angles θ. We have seen, however, that the temperature increases with depth and therefore the rays which propagate closer to the vertical will come from deeper, and hence hotter layers of the atmosphere. Since hotter bodies emit more intense radiation, $I(\theta, 0)$ will have a maximum at $\theta = 0$, which is in agreement with the result we have obtained. The relative decrease of $I(\theta, 0)$ with θ is given by the expression,

$$\frac{I(\theta, 0)}{I(0, 0)} = \frac{2}{5} + \frac{3}{5}\cos\theta = 1 - 0.6 + 0.6\cos\theta \qquad \text{(A-36)}$$

which, as seen in Section 4.2, is in very good agreement with experimental observations of the limb-darkening effect of the solar photosphere.

Radiative Transfer in Radio Astronomy

When a source of intensity I'_ν is behind an absorbing region of opacity τ_ν, the equation of radiative transfer (A-14) in this region and in the line of sight ($\theta = 0$) becomes,

$$\frac{dI_\nu}{d\tau_\nu} = I_\nu \qquad \text{(A-37)}$$

which has the solution,

$$I_\nu = I'_\nu e^{-\tau_\nu} \qquad \text{(A-38)}$$

In the more general case, in computing I_ν we must include also the emission of the absorbing layer. For some generality we can assume that this region is a uniform mixture of thermal plasma and of non-thermal radio sources. Considering only normal incidence ($\theta = 0$), (A-11) in this case takes the form,

$$\frac{dI_\nu}{d\tau_\nu} = I_\nu - \frac{j_\nu}{\varkappa_\nu} - \frac{j''_\nu}{\varrho\varkappa_\nu} \qquad \text{(A-39)}$$

where $\varkappa_\nu$ and j_ν are the free-free mass absorption and mass emission coefficients of the thermal plasma and j''_ν the volume emission coefficient of the non-thermal radiation. The solution of this differential equation is,

$$I_\nu = I'_\nu e^{-\tau_\nu} + \int_0^{\tau_\nu} \left(\frac{j_\nu}{\varkappa_\nu} - \frac{j''_\nu}{\varrho\varkappa_\nu}\right) e^{-\tau'_\nu}\, d\tau'_\nu \qquad \text{(A-40)}$$

where τ_ν is, as we mentioned above, the opacity of this region. In such problems we usually assume a uniform temperature T_0 for the entire plasma region. In this case j_ν, j''_ν, and $\varkappa_\nu$ are independent of τ_ν and can be taken out of the integral to give,

$$I_\nu = I'_\nu e^{-\tau_\nu} + \left(\frac{j_\nu}{\varkappa_\nu} - \frac{j''_\nu}{\varkappa_\nu}\right)(1 - e^{-\tau_\nu}) \qquad \text{(A-41)}$$

For a thermal plasma $j_\nu/\varkappa_\nu = B_\nu(T_0)$, and for the non-thermal emission we have,

$$\frac{j''_\nu}{\varrho\varkappa_\nu} = \frac{j''_\nu L}{\varrho\varkappa_\nu L} = \frac{I''_\nu}{\tau_\nu} \qquad \text{(A-42)}$$

where L is the thickness of this region. Hence (A-41) can be written in the form,

$$I_\nu = I'_\nu e^{-\tau_\nu} + \left[B_\nu(T_0) + \frac{I''_\nu}{\tau_\nu}\right](1 - e^{-\tau_\nu}) \qquad \text{(A-43)}$$

In radioastronomical problems we have, as a rule, that $h\nu/kT \ll 1$, and therefore (A-13) is simplified to the Rayleigh-Jeans formula,

$$B_\nu(T_0)\, d\nu = \frac{2kT_0}{\lambda^2}\, d\nu \qquad \text{(A-44)}$$

Using similar formulas to relate I_ν, I'_ν, and I''_ν to their equivalent temperatures T_b, T', and T'', we can convert (A-43) to a relation of temperatures,

$$T_b = T' e^{-\tau_\nu} + \left(T_0 + \frac{T''}{\tau_\nu}\right)(1 - e^{-\tau_\nu}) \qquad \text{(A-45)}$$

where T_b is the *brightness temperature* we observe with our antennas. In the simple case of a thermal plasma region (HII-region), without I'_ν or I''_ν, (A-45) becomes,

$$T_b = T_0(1 - e^{-\tau_\nu}) \qquad \text{(A-46)}$$

When τ_ν tends to infinity (optically thick medium) we simply have $T_b = T_0$, while when τ_ν tends to zero (optically thin medium) T_b also tends to zero. An application of this equation in the case of the solar corona is given at the end of Section 4.3.