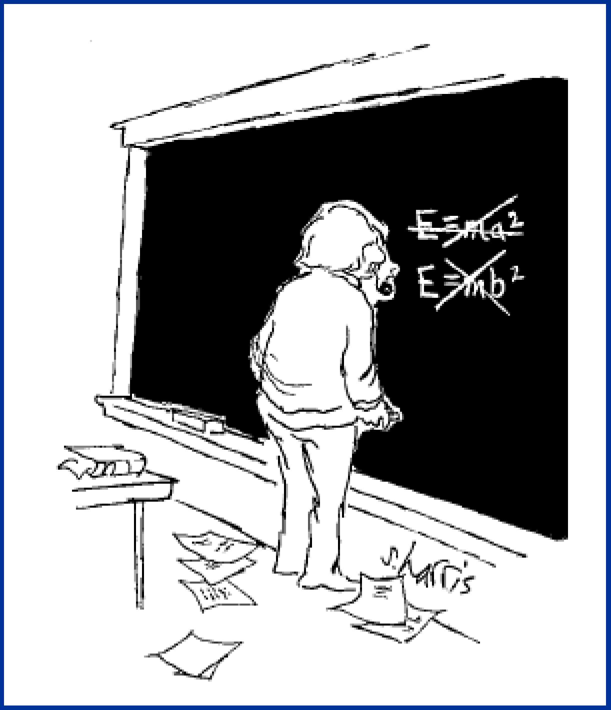
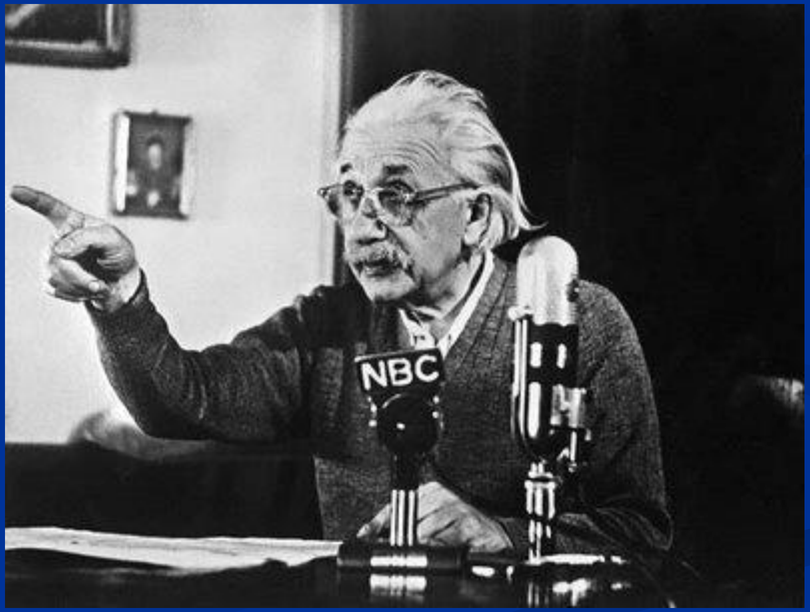
Special Theory of Relativity





3rd Lecture

Special Theory of Relativity

Einstein's Two Postulates in STR

Postulate 01 : The Principle of Relativity:

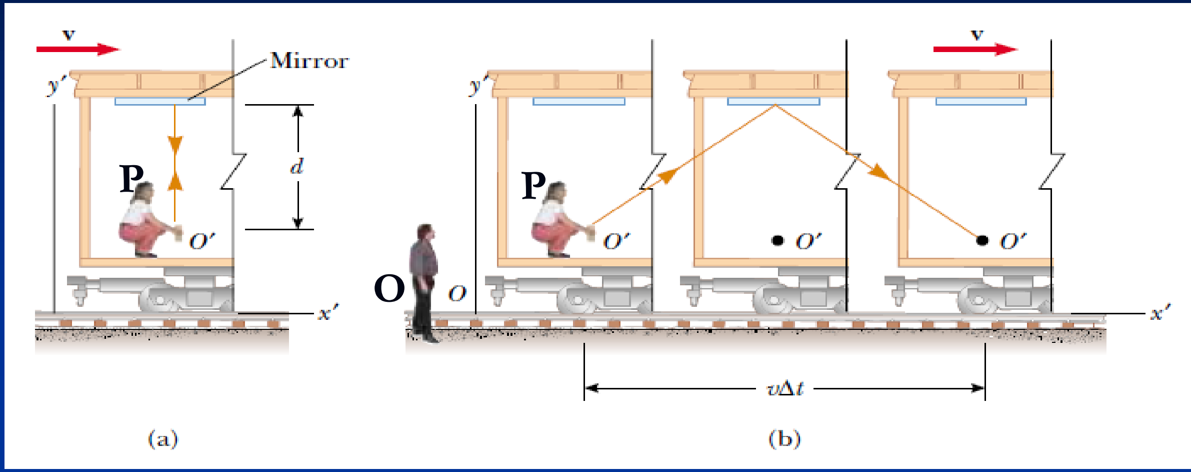
The laws of physics must be the same in all inertial reference frames.

The laws of Physics are the same for all observers in uniform motion relative to one another.

Postulate 02 : The constancy of the speed of light :

The speed of light in vacuum has the same value, c = 3 x 10^8 m/s in all inertial frames, regardless of the velocity of the observer or the velocity of the source emitting the light.

Measurement of Time in STR



$$\Delta t_O = \Delta t_P \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

This equation express the fact that for the moving observer the period of the clock is shorter than in the frame of the ground observer itself !

Where, v is the Relative Speed of the Two Frames

$$\Delta t_O \rangle \Delta t_P$$

Time interval w. r. t the stationary frame

Time interval w. r. t the moving frame

Time Dilation

Suppose, $t_2 = t_1 \dfrac{1}{\sqrt{1 - \dfrac{v^2}{c^2}}}$ This equation is called relativistic time equation!

If v > 0 $\Rightarrow \dfrac{v}{c} < 1 \Rightarrow \dfrac{v^2}{c^2} < 1 \Rightarrow 1 - \dfrac{v^2}{c^2} < 1$

$\Rightarrow \sqrt{1 - \dfrac{v^2}{c^2}} < 1 \Rightarrow t_2 = t_1 \dfrac{1}{(<1)}$

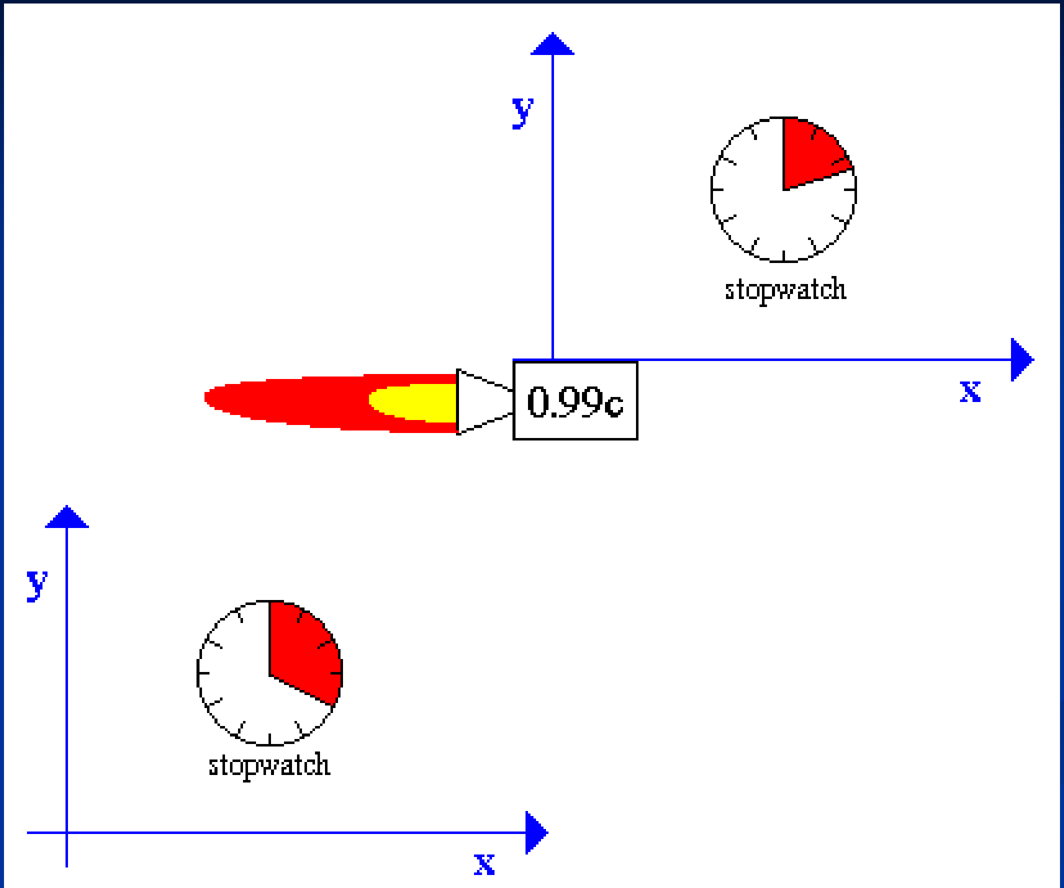
$$t_2 > t_1$$

Time interval w. r. t the stationary frame

Time interval w. r. t the moving frame

This is called Time Dilation !

Time Dilation

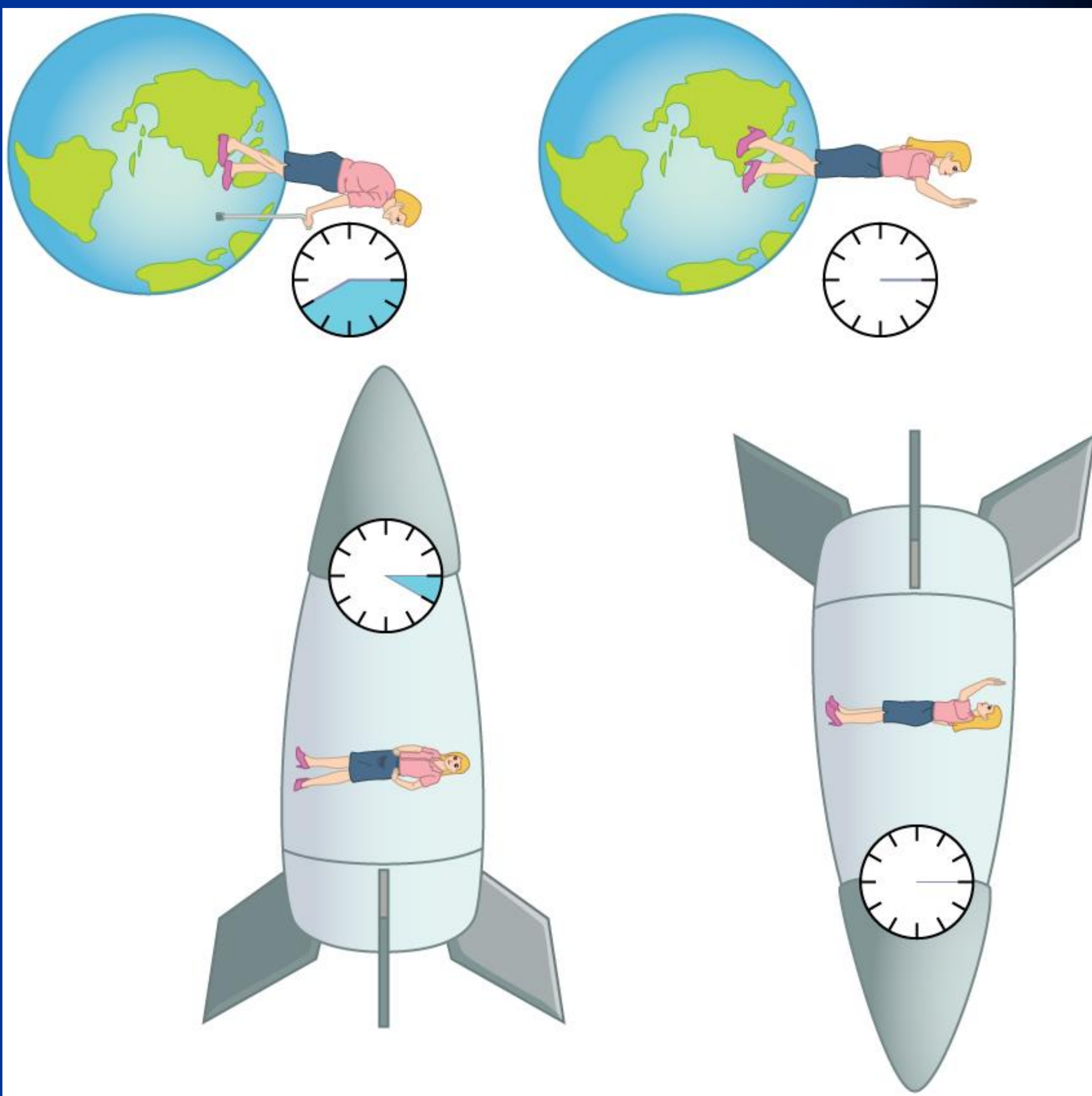


$$t_2 \rangle \; t_1$$

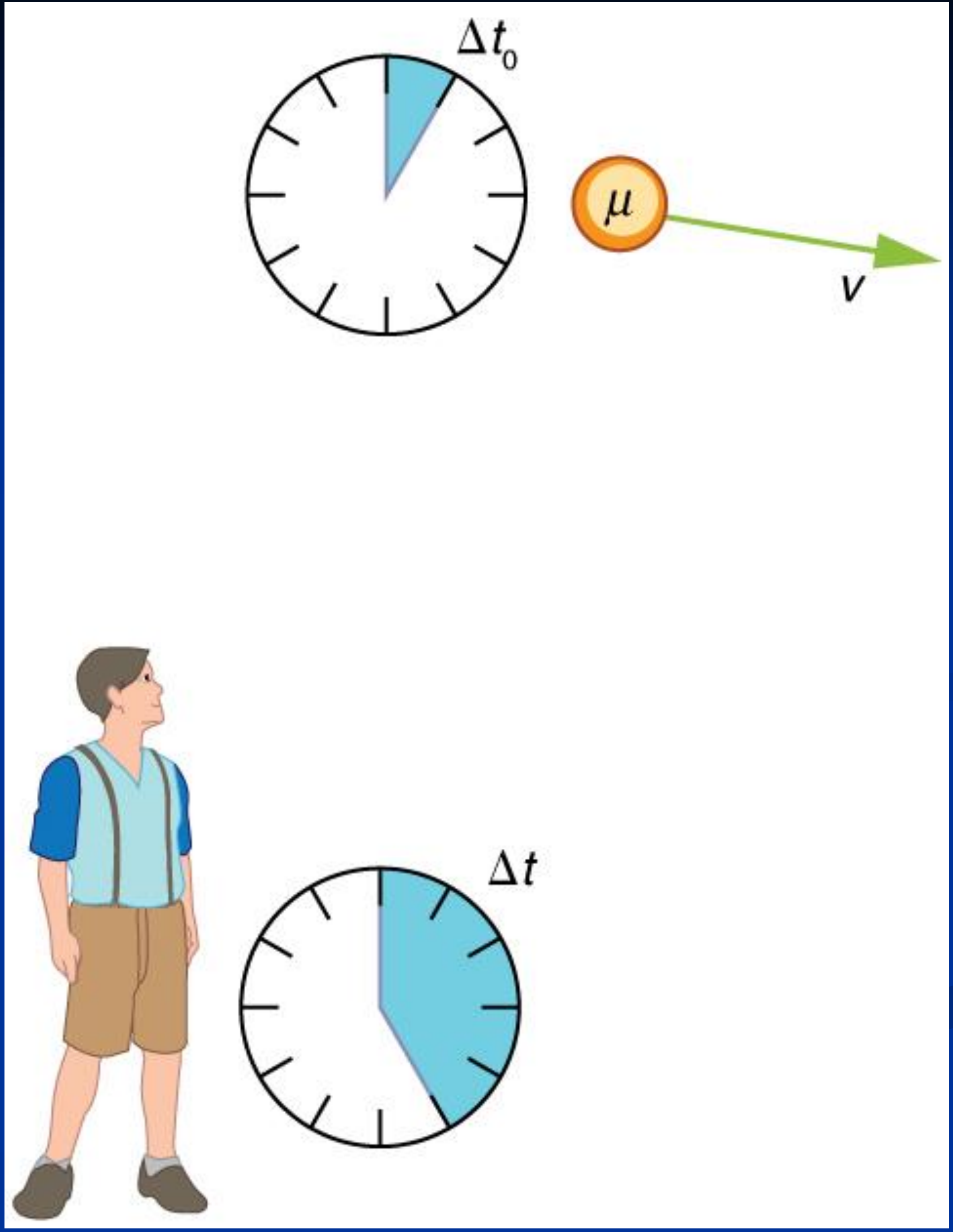
Time interval w. r. t the stationary frame

Time interval w. r. t the moving frame

Time Dilation



Time Dilation



Proper Time & Improper Time

Proper Time : In relativity, proper time is time measured by a single clock between events that occur at the same place as the clock. It depends not only on the events but also on the motion of the clock between the events!

Improper Time : Time measured with two clocks or a single moving clock!

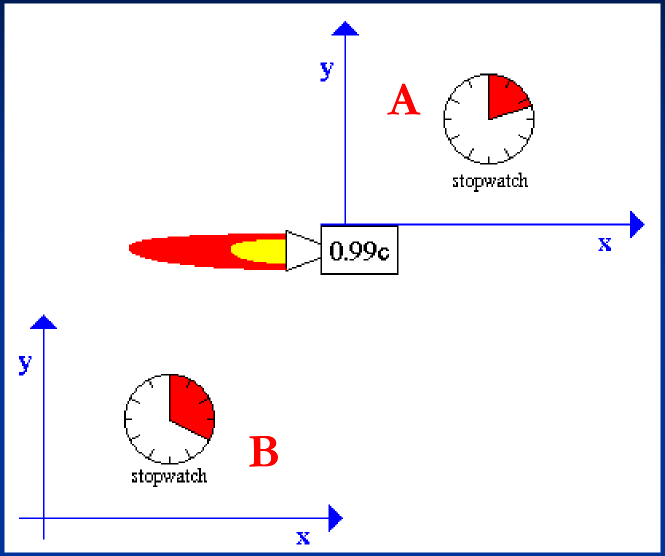
$$\text{Improper Time} = \text{Proper Time} \times \boxed{\sqrt{1-\frac{v^2}{c^2}}}$$

This means, that time is dilated (extended) in motion. This gives rise to two questions,

(01) Do moving clocks really run slow ???

(02) The clock mechanism affected by motion ???

Proper Time & Improper Time



The above results tells us that " *moving clocks run slow* ". When the clock is moving one is measuring improper time. A stationary clock measures proper time. It seems as through there is a Clock Paradox in question **01** started above. For instance, suppose an observer A is moving with a uniform velocity and another observer B is "stationary". Then according to A his watch runs slower than that of B, as a consequence of measurement of proper and improper time !

Question **02** started above, regarding whether the clock mechanism is affected during motion is simply non-sense. As our frames are inertial all physical laws, including mechanics remain unaltered.

Example : A particle X, which is created in a particle accelerator, travels a total distance of $100.0\ m$ between two detectors in $410\ ns$ as measured in the laboratory frame before decaying into other particles. What is the lifetime of the particle X as measured in its own frame ???

Velocity of the particle X w.r.t lab frame : $\boxed{v = \dfrac{distance}{time}}$

$$\boxed{v = \frac{100\,m}{410\,ns}}$$

$$\boxed{v = 2.44 \times 10^{8}\ ms^{-1}}$$

Using relativistic time equation : $\boxed{t_2 = t_1 \dfrac{1}{\sqrt{1-\dfrac{v^2}{c^2}}}}$

$\boxed{t_1 = \ ?}$

$\boxed{t_2 = 410 \times 10^{-9}\ s}$

$\boxed{v = 2.44 \times 10^{8}\ ms^{-1}}$

$$\boxed{t_1 = t_2 \sqrt{1-\frac{v^2}{c^2}}} \Rightarrow \boxed{t_1 = 410 \sqrt{1-\frac{(2.44\times10^{8})^2}{(3.0\times10^{8})^2}}} \Rightarrow \boxed{t_1 = 238\,ns}$$

Experiment on Time Dilation :

The experimental evidence for time dilation was provided by Ives and Stilwel (1938) who measured the change in frequency of spectral lines emitted by fast moving atoms. The effect observed was small as the velocities of atoms was only about c/2; but it was convincing.

The real, conclusive evidence came from the experiment of Rossi & Hall (1941) involving **muons**, and we discuss this below :

The muon is a charged particle with mass about 200 times that of an electron. It decays to an electron plus a neutrino – antineutrino pair.

$$\mu \rightarrow e + \gamma_{\mu} + \overline{\gamma}_{\mu}$$

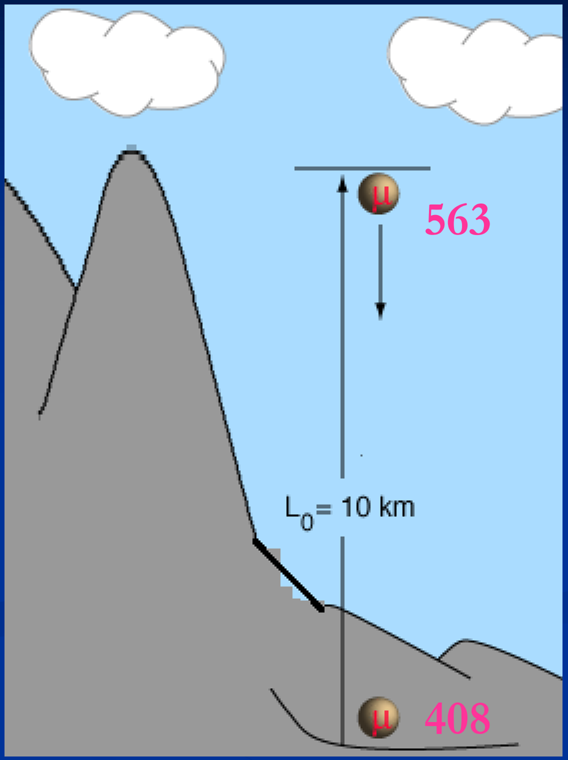
Antineutrino

Neutrino

and the half life for the muon decay is *1.53 x 10^(-6) s*. (This means that if there are N_o number of muons at time t = 0, then after *1.53 x 10^(-6) s*, there will be $N_o/2$ muons left. The rest $N_o/2$ would have decayed.)

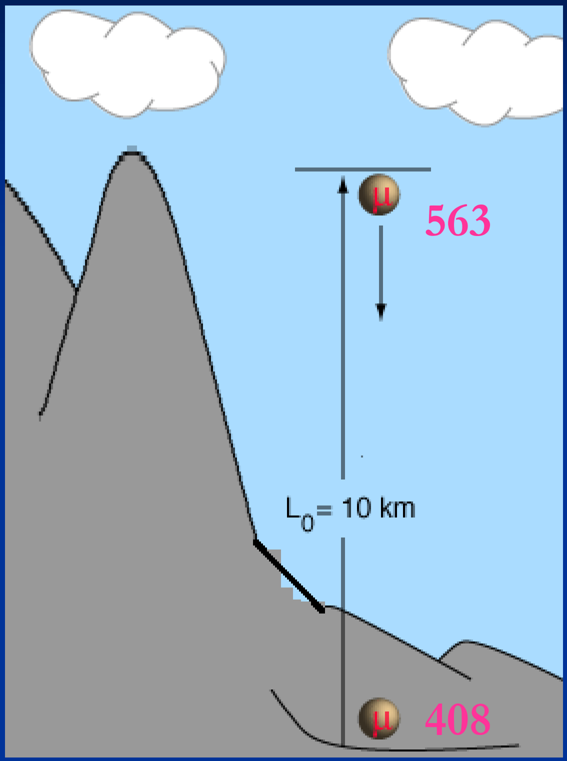
Experiment on Time Dilation…

Cosmic rays contain large number of muons at high altitudes and they mostly travel vertically downward at speeds comparable with that of light. The experiment of Rossi & Hall consisted in measuring the number of muons and their time of flight at the top of **Mountain Washington** (height 6265 ft) in New Hampshire, and at 10 ft above sea level. Only those muons with speed between 0.9950 c and 0.9954 c were counted. This number at the mountain top was found to be 563±10 per hour.



However, at sea level the number of muons was found to be 408±9 per hour. **It must be said that in the experiment Rossi & Hall did not detect the 408±9 muons remaining out of the 563±10 per hour**. The ground experiment was done at some outer place which was close to the mountain.

Experiment on Time Dilation…



Thus, the measured time of the muon flight, which is the improper time is,

$$t_{im} = \frac{(6265 - 10)\,ft}{0.9952 \times 3 \times 10^{8} \times 3.28\ (ft/m)}$$

$$\Rightarrow \quad t_{im} = 6.4 \times 10^{-6}\, s$$

This experiment verifies that,

$$t_{improper} \rangle\ t_{proper}$$

This experiment clearly shows that time dilation can be a very significant effect for clocks that are in high speed relative motion!

$$t_A = t_B \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

```
In[1]:= tb = 10; (* Time in Seconds *)
        c = 3*10^(8); (* Velocity of Light *)
        v = 60; (* Speed of the vehicle *)
        ta = N[tb*(1/Sqrt[1 - v^2/c^2]), 50] (* Time in Seconds *)

Out[3]= 10.000000000000200000000000000600000000000002000000000
```

13th Decimal Point

```
In[4]:= tb = 10; (* Time in Seconds *)
        c = 3*10^(8); (* Velocity of Light *)
        v = c/10; (* Speed of the particle *)
        ta = N[tb*(1/Sqrt[1 - v^2/c^2]), 50] (* Time in Seconds *)

Out[5]= 10.050378152592120754893735565668747527051783471483
```

2nd Decimal Point

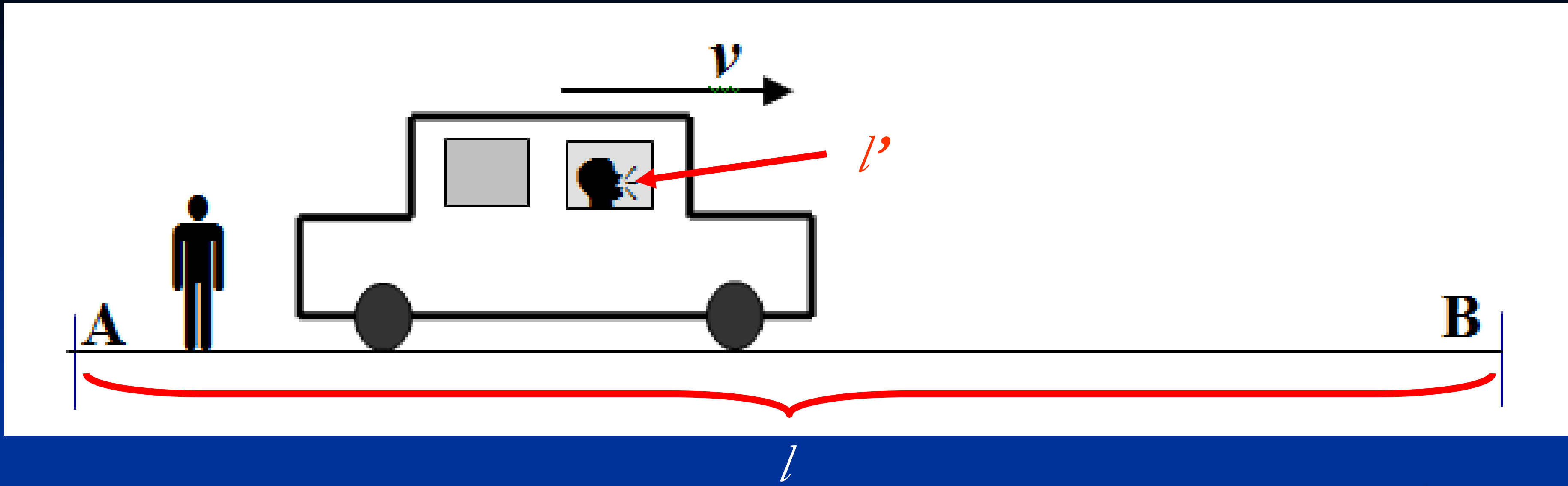
```
In[6]:= tb = 10; (* Time in Seconds *)
        c = 3*10^(8); (* Velocity of Light *)
        v = c/2; (* Speed of the particle *)
        ta = N[tb*(1/Sqrt[1 - v^2/c^2]), 50] (* Time in Seconds *)

Out[7]= 11.547005383792515290182975610039149112952035025403
```

Measurement of Length in STR

So, time is relative ! What about distance ???

In order to think about this note that when we say that the distance between two objects is l we imagine measuring the position of these objects simultaneously… but simultaneously is relative, so **we can expect distance to be a relative concept too**!

Length contraction is the observation that a moving object appears shorter than a stationary object! Like time dilation, length contraction is a consequence of the postulates of relativity. Length contraction and time dilation are related, and introduce a relation between length and time (Or space and time, if you prefer!)

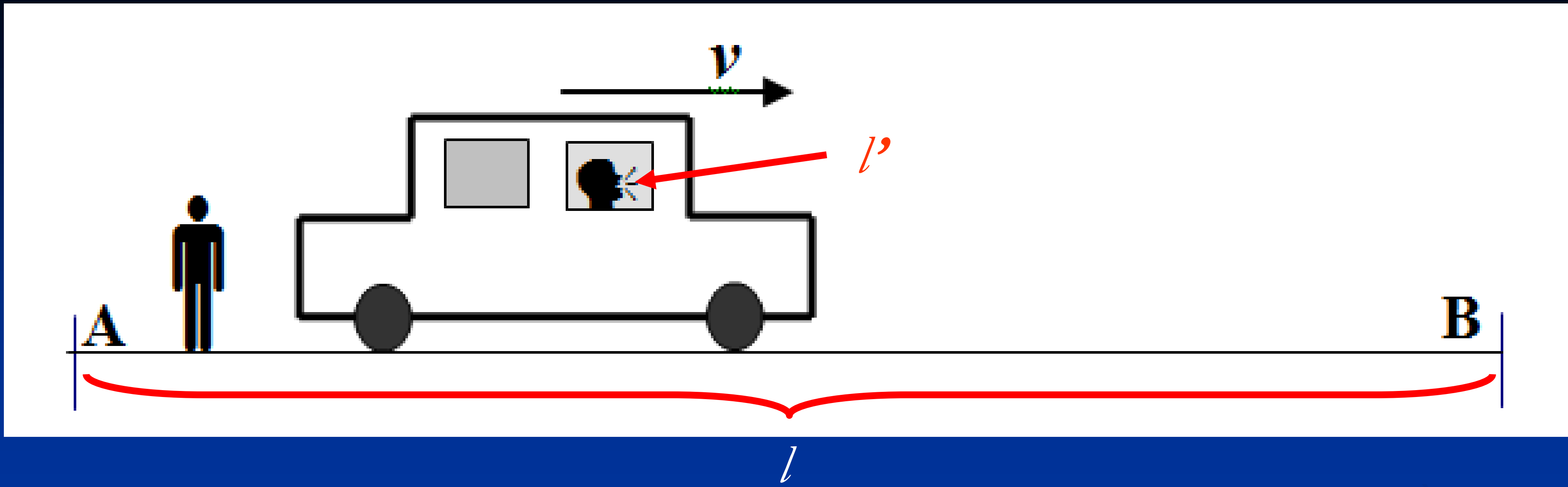


Suppose a car is moving with a uniform speed v relative to an observer in a stationary frame. This observer notes that it covers a distance l between two check posts (A & B) in time t. As two different clocks (one at each post) are involved in the measurement of this time, this time is improper time.

$$t_{im} = \frac{l}{v}$$

The Observer in the car covers the distance between the check-posts in proper time, such that:

$$t_{im} = t_{pro} \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

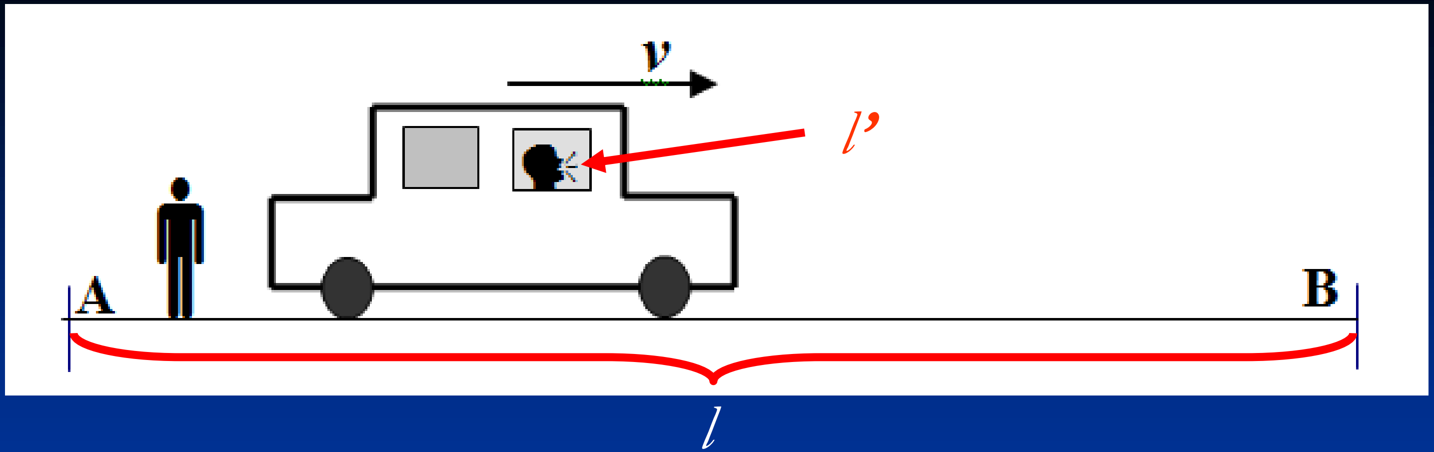


Thus, the distance l' that the observer in the car measures is given by,

$$t_{pro} = \frac{l^1}{v}$$

(l' is measured by using **speedometer in the car** !)

$$t_{im} = t_{pro} \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \Rightarrow \frac{l}{v} = \frac{l^1}{v} \times \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \Rightarrow l^1 = l\sqrt{1-\frac{v^2}{c^2}}$$



$$l^1 = l\sqrt{1-\frac{v^2}{c^2}}$$

This equation is called relativistic length equation!

If v > 0 $\Rightarrow$ $\frac{v}{c} < 1$ $\Rightarrow$ $\frac{v^2}{c^2} < 1$ $\Rightarrow$ $1-\frac{v^2}{c^2} < 1$

$\Rightarrow$ $\sqrt{1-\frac{v^2}{c^2}} < 1$ $\Rightarrow$ $l^1 = l\ (<1\ )$

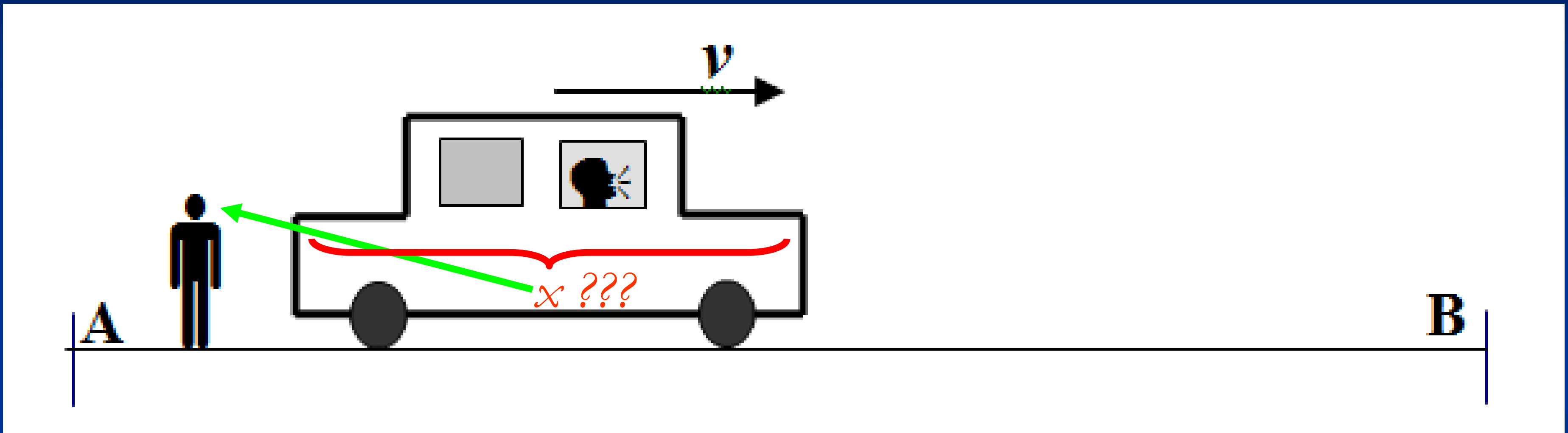
$\Rightarrow$ $l^1 < l$

Length measured by an observer in the car

Length measured by observer on the Earth

This is called Length Contraction !

What is the length of the car as seen by an observer on the Earth ???



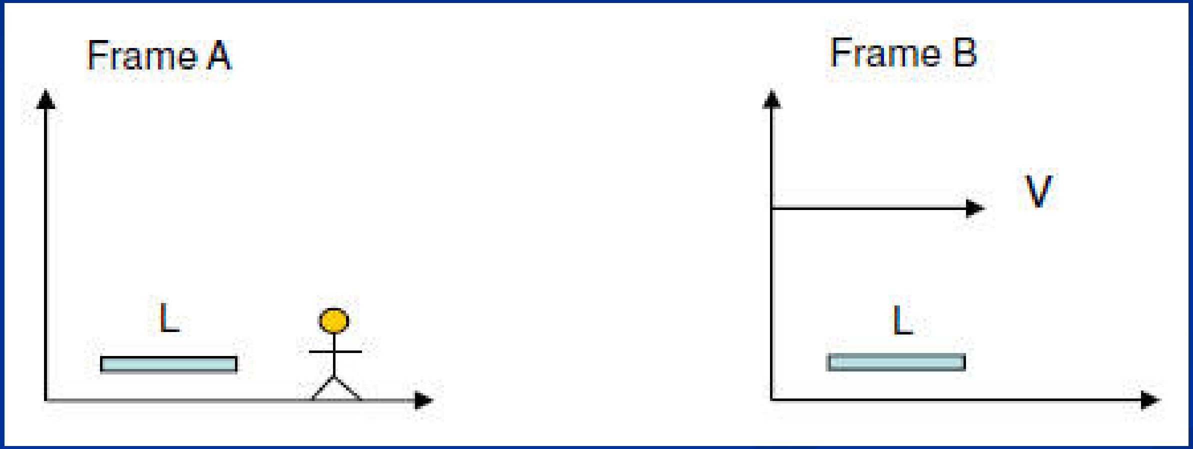
Is it less than the original length ???

Is it greater than the original length ???

Is it equal to the original length ???

Length Contraction

Length contraction is the observation that a moving object appears shorter than a stationary object.



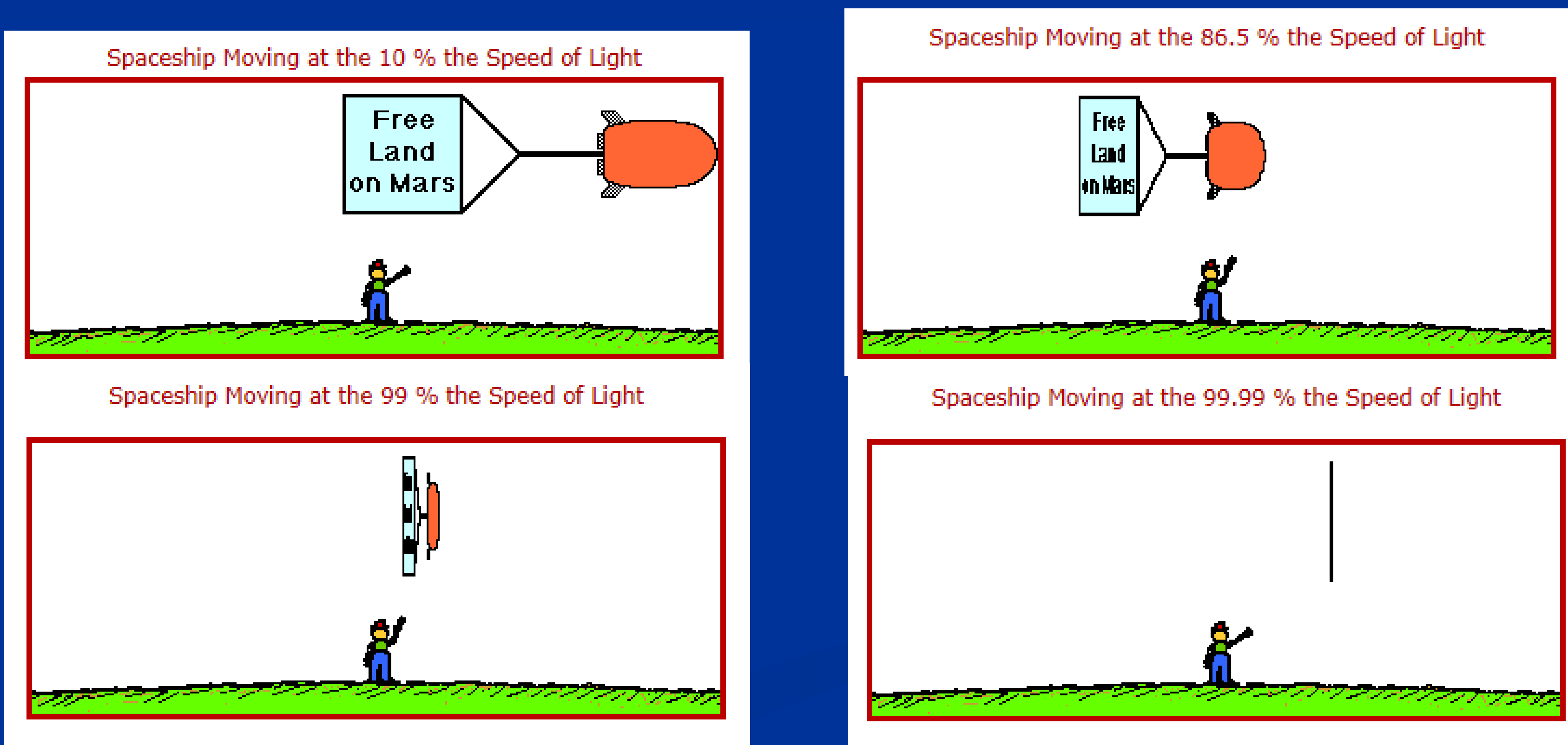
Length contraction – an observer in frame A sees the stick in frame B as shorter than the stick in frame A.

$$l^1 < l$$

Length measured by an observer in the Frame A

Length measured by an observer in the Frame B

One of the peculiar aspects of Einstein's theory of special relativity is that the length of objects moving at relativistic speeds undergoes a contraction along the dimension of motion. An observer at rest (relative to the moving object) would observe the moving object to be shorter in length. That is to say, that an object at rest might be measured to be 200 feet long; yet the same object when moving at relativistic speeds relative to the observer/measurer would have a measured length which is less than 200 ft. This phenomenon is not due to actual errors in measurement or faulty observations. The object is actually contracted in length as seen from the *stationary reference frame*. The amount of contraction of the object is dependent upon the object's speed relative to the observer.



http://www.physicsclassroom.com/mmedia/specrel/lc.cfm

Example :

A *200 m* long train passes through a tunnel *100 m* long with the constant speed $c/4$.

(*a*) What is the length of the tunnel as seen by an observer in the train ???

(*b*) What is the length of the train as seen by an observer on the Earth ???

(*a*)

$$Length\ (\text{Tunnel, E}) = 100\ m$$

$$Length\ (\text{Tunnel, Train}) = l\ ?$$

Using relativistic length equation :

$$l^1 = l\sqrt{1-\frac{v^2}{c^2}}$$

$$l = 100\sqrt{1-\frac{\left(\frac{c}{4}\right)^2}{c^2}} \Rightarrow l = 100 \times 0.9682 \Rightarrow l = 96.82\, m$$

Example :

A *200 m* long train passes through a tunnel *100 m* long with the constant speed $c/4$.

(*a*) What is the length of the tunnel as seen by an observer in the train ???

(*b*) What is the length of the train as seen by an observer on the Earth ???

(*b*)

$$Length\ (\text{Train, E}) = 200\ m \quad at\ rest$$

$$Length\ (\text{Train, E}) = l\ ? \quad is\ moving$$

Using relativistic length equation :

$$l^1 = l\sqrt{1-\frac{v^2}{c^2}}$$

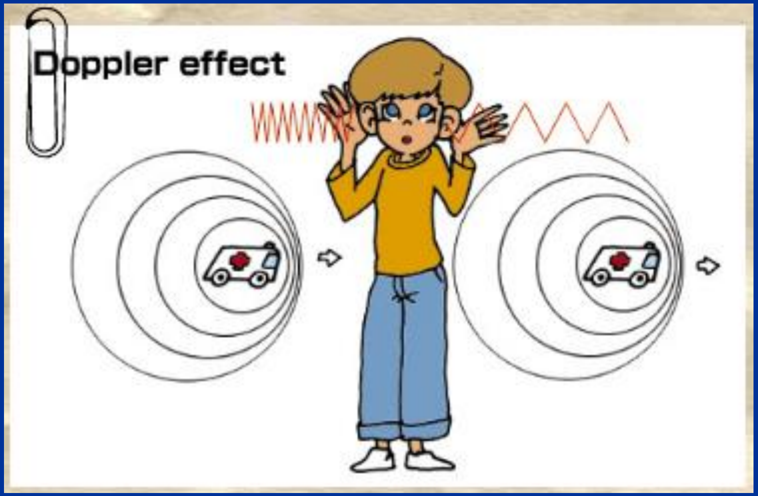
$$l = 200\sqrt{1-\frac{\left(\frac{c}{4}\right)^2}{c^2}} \Rightarrow l = 200 \times 0.9682 \Rightarrow l = 193.64\,m$$

Doppler's Effect in STR

Doppler's Effect for Sound Waves :

The Doppler's effect (or Doppler's shift), named after Austrian physicist Christian Doppler who proposed it in 1842, is the change in frequency of a wave for an observer moving relative to the source of the wave.

It is commonly heard when a vehicle sounding a siren or horn approaches, passes and recedes from the observer.

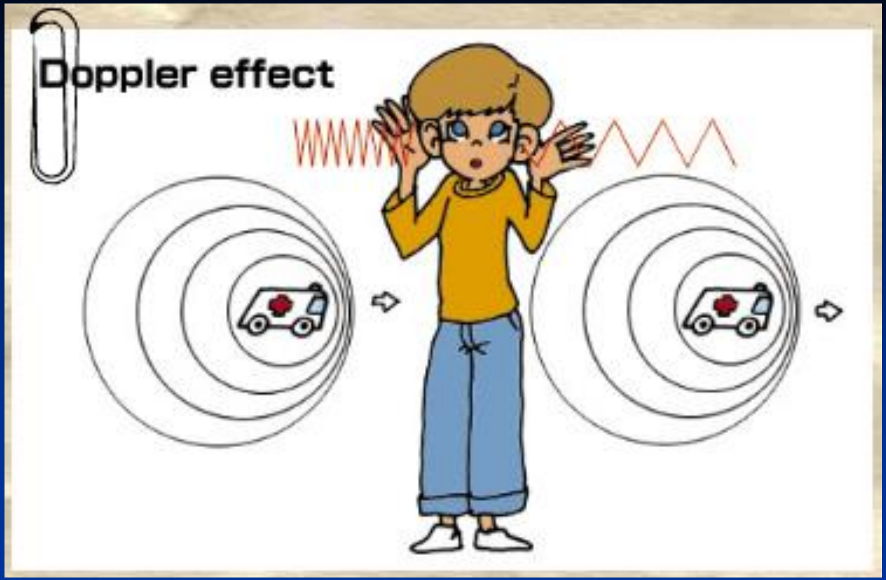


In classical physics (wave in a medium), where the source and the receiver velocities are not supersonic, the relationship between observed frequency f_o and emitted frequency (or source frequency) f_s is given by,

P.T.O

Doppler's Effect for Sound Waves :

$$f_o = f_s\left(\frac{v \pm v_1}{v \mp v_2}\right)$$



Where,

v is the velocity of waves in the medium.

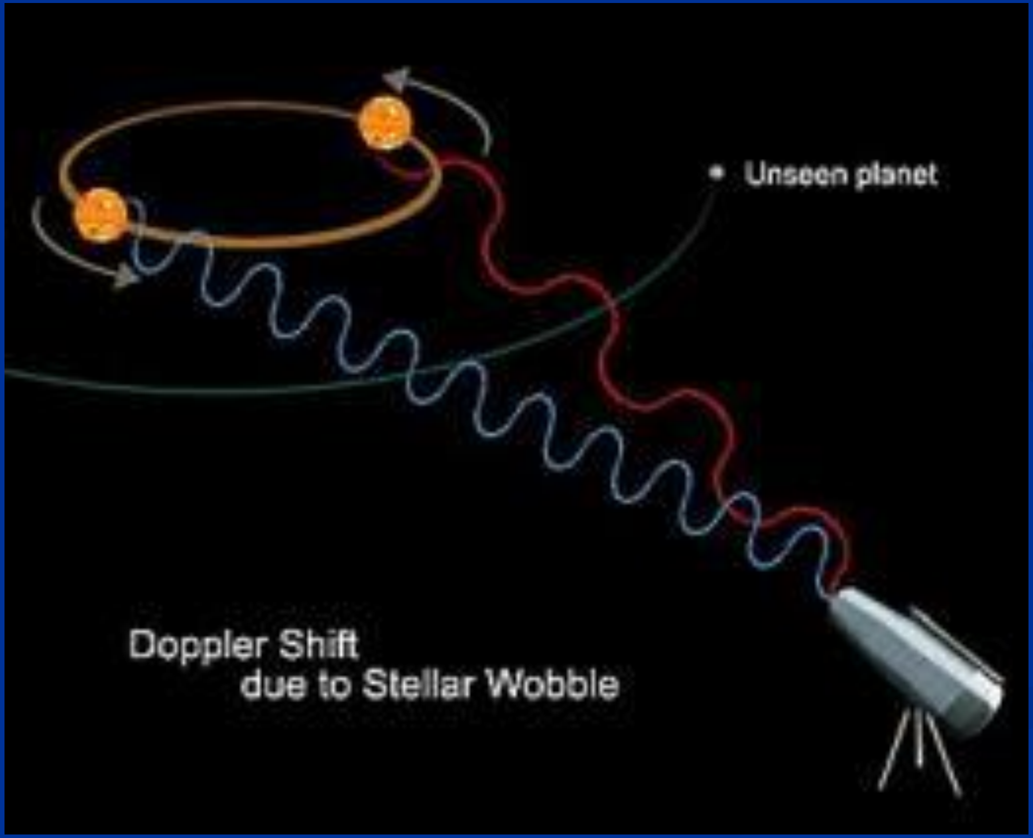
v1 is the velocity of the receiver relative to the medium; positive if the receiver is moving towards the source.

v2 is the velocity of the source relative to the medium; positive if the source is moving away from the receiver.

The frequency is decreased if either is moving away from the other!

Doppler's Effect in STR

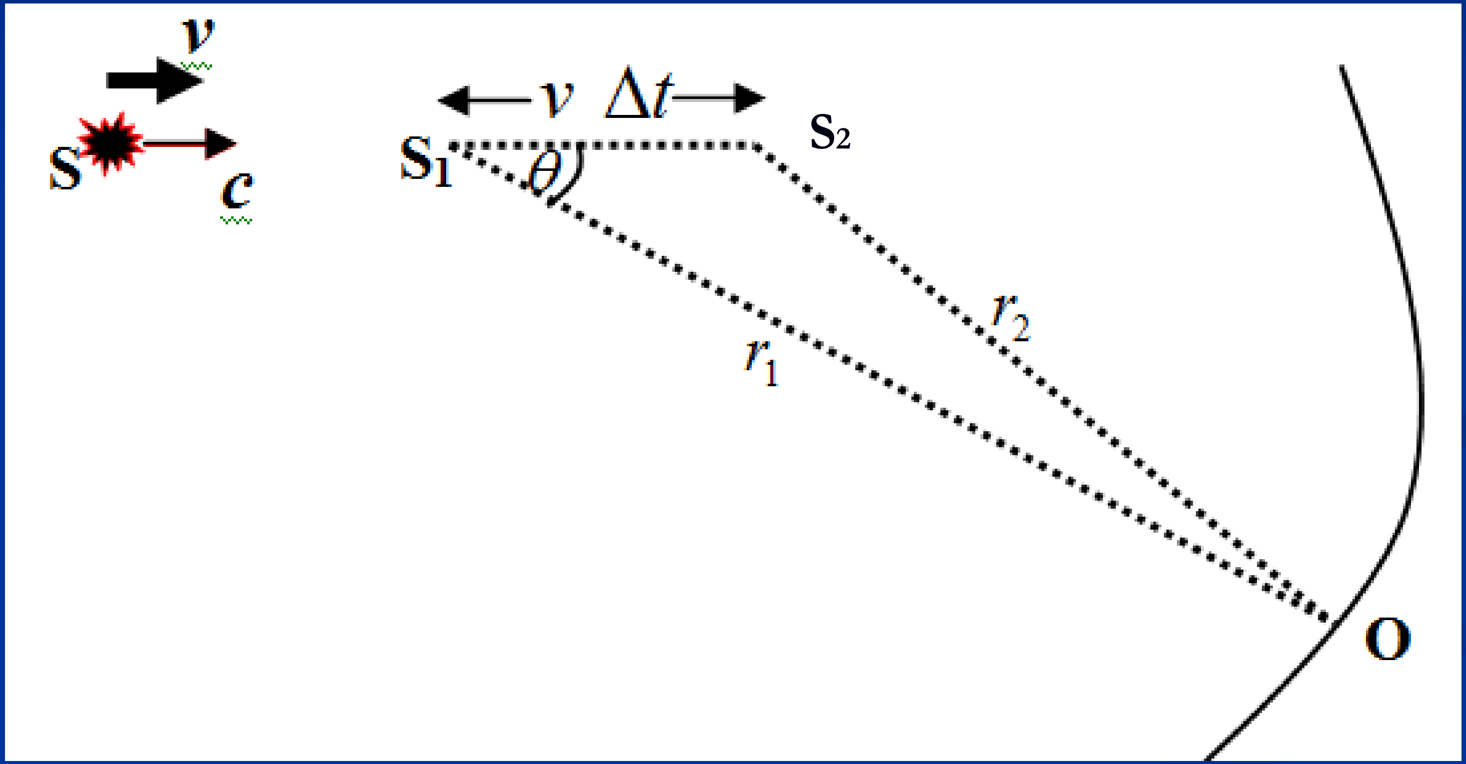
The relativistic Doppler's effect is the change in frequency (and the wavelength) of light, caused by the relative motion of the source and the observer (as in the classical Doppler's effect), when taking into account effects of the Special Theory of Relativity!



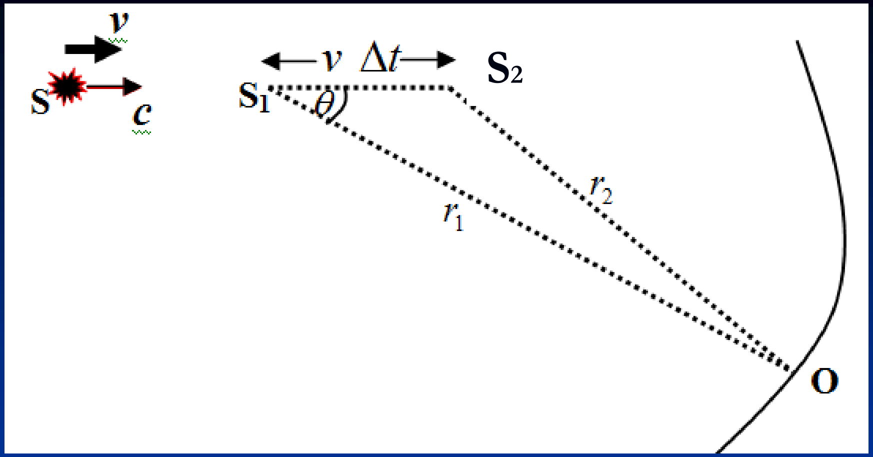
If a light source in uniform motion approaches or recedes a stationary observer then the frequency of light is observed to change. This is known as **Doppler effect** in STR.

P.T.O

We assume a frame in which the source is moving and observer is stationary. Consider a source S of frequency f_s moving with velocity v. The source emits periodic waves. Suppose that when the source is at S_1 it emits a signal. This reaches the observer O after a time r_1/c.



When the signal reaches O the source has moved S_2. Suppose the time taken for the source to move from S_1 to S_2 is Δt. Then $S_1S_2 = v\,\Delta t$.



$$S_1S_2 = v\Delta t$$

This is the distance covered by the source before giving out the next signal. When the source is at S_2 it gives out a signal which is received by the observer at O.

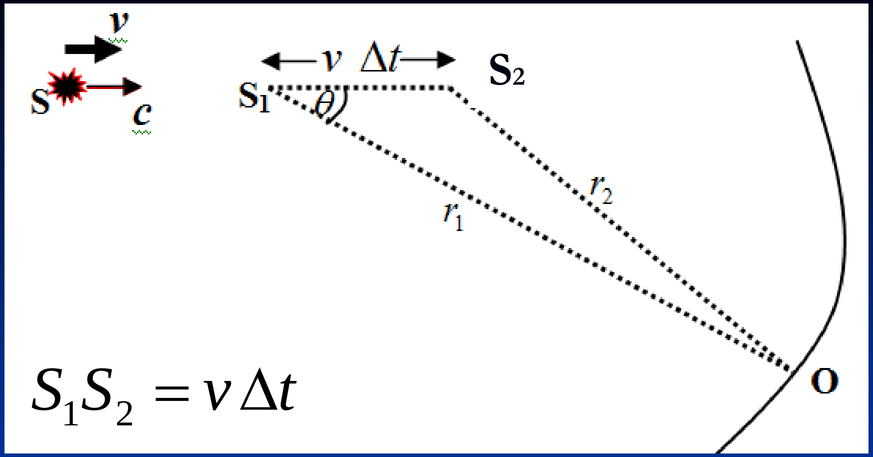
Thus, the time for the second signal to be received at O is, $\Delta t + \frac{r_2}{c}$

The difference in time for the two signals to reach O is then,

$$\Delta t_o = \Delta t + \frac{r_2}{c} - \frac{r_1}{c}$$

$$\Rightarrow \Delta t_o = \Delta t + \frac{(r_2 - r_1)}{c}$$

$\longrightarrow$ 01



From the figure we note that, (Using Cosine Theorem)

$$r_2^2 = r_1^2 + (v\Delta t)^2 - 2r_1(v\Delta t)\cos\theta$$

As $r_1 >> v\Delta t$, we get , $v\Delta t \approx (v\Delta t)\cos\theta$

$$\Rightarrow \quad r_2^2 = r_1^2 - 2r_1(v\Delta t)\cos\theta + ((v\Delta t)\cos\theta)^2$$

$$\Rightarrow \quad r_2^2 = (r_1 - (v\Delta t)\cos\theta)^2 \quad \Rightarrow \quad r_2 = r_1 - (v\Delta t)\cos\theta$$

$$\Rightarrow \quad r_2 - r_1 = -(v\Delta t)\cos\theta$$

→ 02

Connect equation 01 and 02 :

$$\Delta t_o = \Delta t + \frac{(r_2 - r_1)}{c} \Rightarrow \Delta t_o = \Delta t + \frac{(-(v\Delta t)\cos\theta)}{c}$$

$$\Rightarrow \Delta t_o = \Delta t\left(1 - \frac{v\cos\theta}{c}\right) \longrightarrow 03$$

This is the time difference between two successive signals received at O, i.e.: $1 / \Delta t_o$ is the observed frequency of emission.

$$f_o = \frac{1}{\Delta t_o}$$

We now need to connect t_s with the actual frequency of emission by the source, f_s. We realize that $1/f_s$ is the time difference between two successive signals emitted by the source and is therefore a proper time in the frame of the source. On the other hand, Δt is the time interval between the emission of two consecutive signals as seen from the stationary frame.

P.T.O

Using the Relativistic Time equation :

$$\Delta t = \Delta t_s \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\Rightarrow \Delta t_s = \Delta t\sqrt{1-\frac{v^2}{c^2}}$$

$$\Rightarrow \frac{1}{f_s} = \Delta t\sqrt{1-\frac{v^2}{c^2}}$$

Using equation 03 ;

$$\Delta t_o = \Delta t\left(1-\frac{v\cos\theta}{c}\right)$$

We get,

$$\Delta t = \Delta t_o \frac{1}{\left(1-\frac{v}{c}\cos\theta\right)}$$

Then we have,

$$\frac{1}{f_s} = \Delta t\sqrt{1-\frac{v^2}{c^2}} \Rightarrow \frac{1}{f_s} = \Delta t_o \frac{1}{\left(1-\frac{v}{c}\cos\theta\right)}\sqrt{1-\frac{v^2}{c^2}} \Rightarrow$$

P.T.O

$$\frac{1}{f_s} = \Delta t_o \frac{1}{\left(1 - \frac{v}{c}\cos\theta\right)} \sqrt{1 - \frac{v^2}{c^2}}$$

$$\Rightarrow \quad \frac{1}{f_s} = \Delta t_o \frac{1}{\gamma(1 - \beta\cos\theta)}$$

where, $\gamma = \dfrac{1}{\sqrt{1 - \dfrac{v^2}{c^2}}}$ and, $\beta = \dfrac{v}{c}$

As the frequency f_o observed by the observer is $1/\Delta t_o$, we obtain,

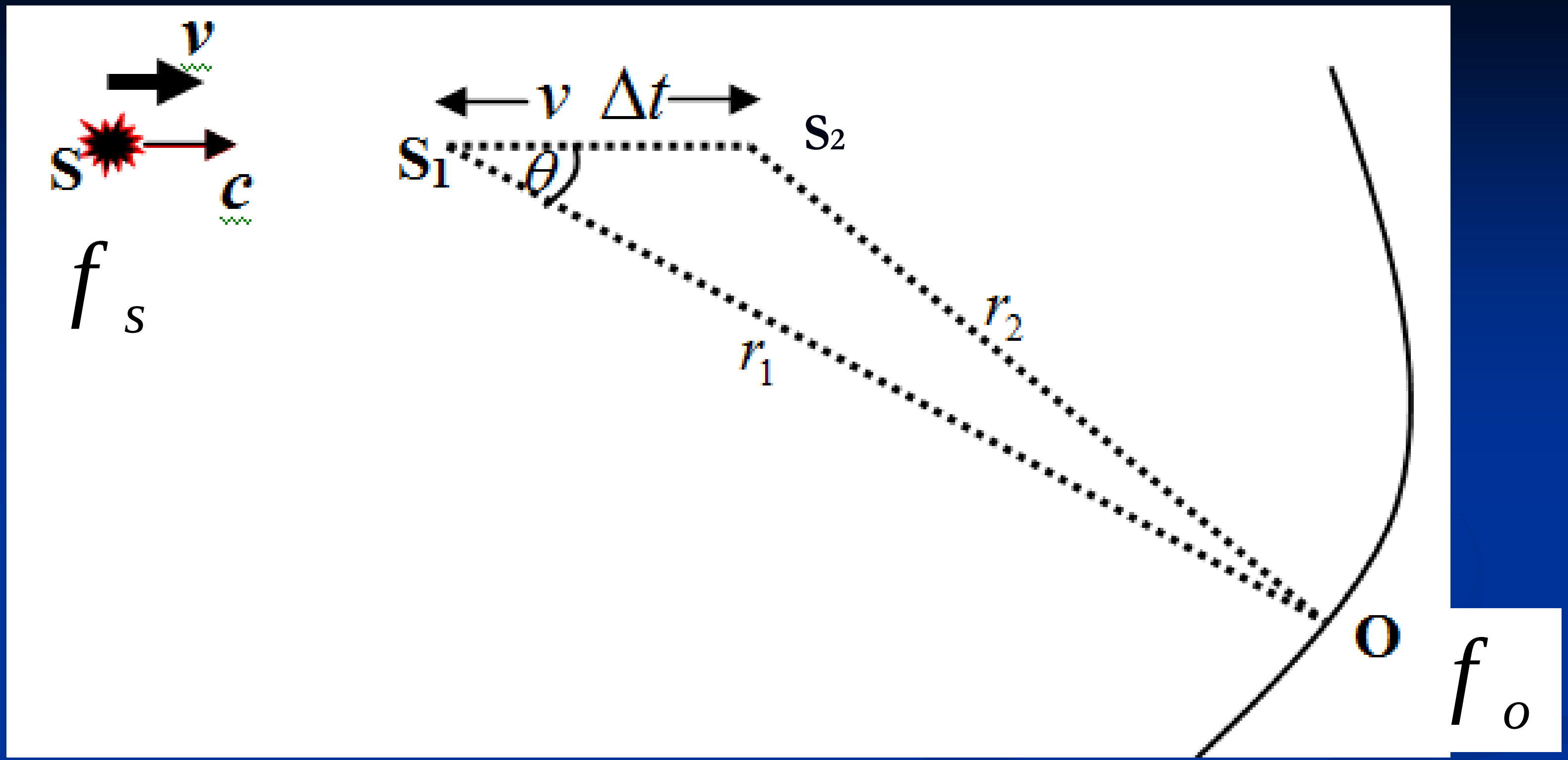
$$f_o = \frac{1}{\Delta t_o}$$

$$\frac{1}{f_s} = \Delta t_o \frac{1}{\gamma(1 - \beta\cos\theta)} \quad \Rightarrow \quad \frac{1}{f_s} = \frac{1}{f_o} \frac{1}{\gamma(1 - \beta\cos\theta)}$$

$$\Rightarrow \quad f_o = f_s \frac{1}{\gamma(1 - \beta\cos\theta)}$$

where, $\gamma = \dfrac{1}{\sqrt{1 - \beta^2}}$ $\quad \beta = \dfrac{v}{c}$

This is the general form of the Doppler's Effect in STR!



$$f_o = f_s \frac{1}{\gamma(1-\beta\cos\theta)}$$

where, $\gamma = \dfrac{1}{\sqrt{1-\beta^2}}$

and $\beta = \dfrac{v}{c}$

This is the general form of the Doppler's Effect in STR!

Thank You !